A Two Dimensional Computer Programme for solving the MHD equations for the Theta Pinch in a time-dependent Coordinate System.

F. Hertweck
W. Schneider

IPP 6/90
IPP 1/110 Juli 1970

INSTITUT FÜR PLASMAPHYSIK
GARCHING BEI MÜNCHEN

INSTITUT FÜR PLASMAPHYSIK

GARCHING BEI MÜNCHEN

A Two Dimensional Computer Programme
for solving the MHD equations for the
Theta Pinch in a time-dependent Coordinate
System.

F. Hertweck
W. Schneider

IPP 6/90
IPP 1/110 Juli 1970

Die nachstehende Arbeit wurde im Rahmen des Vertrages zwischen dem Institut für Plasmaphysik GmbH und der Europäischen Atomgemeinschaft über die Zusammenarbeit auf dem Gebiete der Plasmaphysik durchgeführt.

IPP 6/90
IPP 1/110

F. Hertweck
W. Schneider

A Two Dimensional Computer Programme for solving the MHD equations for the Theta Pinch in a time-dependent Coordinate System.
(in English)

July 1970

Abstract

The axial losses of mass and energy at the ends of a theta pinch are investigated.

The plasma is described by a one-fluid model with the following assumptions: infinite electrical conductivity ($\sigma = \infty$), electron temperature equal to ion temperature, isotropic pressure, and radial equilibrium.

As boundary conditions for the magnetic field $\vec{B}$ it is assumed that the current in the coil is constant in time and that $\vec{B}$ is continued periodically at the end plane. For the dynamic quantities ϱ, $(\varrho \cdot \vec{v})$, P and the heat flow $\vec{q}$ boundary conditions are used which allow free outflow at the end plane. The initial condition corresponds to an already compressed plasma in radial equilibrium.

The resulting system of partial differential equations for the quantities ϱ, $\vec{v}$, p and $\vec{B}$ as functions of r, z and t is solved numerically in magnetic field line coordinates. The main advantage of this coordinate system is the more exact computation of effects parallel to the magnetic field lines, because there is no numerical diffusion.

It is investigated how the disturbance due to mass and energy losses at the end of the theta pinch propagates into the inner part of the vessel. The relative importance of kinetic energy, heat conduction, and convection for the end losses is calculated.

This programme is intended as a first step towards a more sophisticated programme with a two-fluid model including anisotropic pressure.

I N D E X

Introduction

The escape of the plasma from the ends of a theta pinch is one of the main obstacles to confining a high temperature plasma for a sufficiently long time in such an experiment.

As a contribution to a fluid theory for the plasma, this paper is therefore mainly intended for describing this escape quantitatively, i.e. for determining the parameters ϱ, $\vec{v}$, p, and $\vec{B}$ as functions of r, z, and t and hence, in particular, the end losses of mass and energy.

The first stage in the development of such a two-dimensional computer programme is provided by a relatively simple plasma model, viz. the one-fluid model (ion and electron temperatures equal) with isotropic pressure $P_{ik} = p\,\delta_{ik}$.

The complicated compression processes (investigated in one-dimensional programmes by D.Düchs /5/ and H.Fisser /6/ are not treated in these calculations. The main purpose of the programme is to investigate the dynamics in the z-direction at times after compression when the effects of radial processes are less pronounced.

The assumption of infinite electrical conductivity ($\sigma = \infty$), i.e. an infinitely thin current carrying sheath, is not so severe a restriction on the plasma at these relatively late times as it may seem at first glance.

Furthermore, neglecting the inertia terms in the radial direction, i.e. eliminating the oscillations in the r-direction and hence the description of the r-dependences by radial equilibrium, is admittedly a gross simplification, but as the dynamics in the z-direction is our chief interest this approach is adequate as a first approximation.

The boundary conditions are particularly important for the problem. Actually, the entire external space ought to be calculated as well in order to describe exactly the distribution of the magnetic field and the reaction of the escaping plasma on the interior of the coil. This is virtually impossible in view of the enormous computing time required. Since, however, this reaction fortunately does not have much effect, a periodic continuation was assumed for the magnetic field and a free escape into the vacuum for the dynamic quantities. This adequately simulates the actual conditions.

The system of coupled differential equations for this model is treated in a special coordinate system in which the surfaces r = const of the cylindrical coordinates are replaced by surfaces of constant magnetic flux. These curvilinear, non-orthogonal, and time dependent magnetic field coordinates are particularly suitable for treating the problem numerically.

The solutions $Y = Y(r,z,t)$, $Y : \{\varrho, \vec{v}, p, \vec{B}\}$, of the system of differential equations show that the outflow is reduced by introducing thermal conduction. This means that there is a decline in the mass losses and the energy losses due to convection and kinetic energy decrease. However, the total energy and the local thermal energy decrease faster with time in the case with thermal conduction since the loss mechanism due to diffusion has to be taken into account as well. At later times (when the total energy of the plasma is down to about 1o %) there is no difference in the total energy between the cases with and without thermal conduction.

The results of calculations for the ISAR III theta pinch experiment are compared with the measurements of end losses in this device.

1. Fundamental equations of the model

In order to study a two-dimensional programme in a relatively simple model first, only the most simple fundamental equations were used for a start, viz. a one-fluid model, i.e. a fully ionized plasma with equal ion and electron temperatures.

The following assumptions are made for the fully ionized plasma:

a. $\eta = 0$ infinite electrical conductivity

b. $T^{(i)} = T^{(e)}$ (hence $p^{(i)} = p^{(e)}$ owing to quasi-neutrality)

c. $P_{\alpha\beta} = p\,\delta_{\alpha\beta}$ isotropic pressure.

This yields a simplified system of equations, viz.

continuity equations of the entire plasma
equation of motion of the entire plasma,
pressure equation of the entire plasma (electrons and ions).

All collision terms vanish.

In generalized Ohm's law the pressure gradients are also neglected in addition to the inertia terms and the Hall term $\vec{j} \times \vec{B}$. Since infinite electrical conductivity ($\sigma = \infty$), in particular, is assumed in our case, i.e. $\eta = 0$, Ohm's law is of the form:

$$\vec{E} + \vec{V} \times \vec{B} = 0.$$

1.1 Transport coefficients - thermal conductivity

As this paper deals only with one pressure equation obtained by adding the equations for the ion and electron pressures, the coefficient of thermal conductivity is

$$\varkappa = \varkappa^{(i)} + \varkappa^{(e)}$$

The thermal conductivities of ions $\varkappa^{(i)}$ and electrons $\varkappa^{(e)}$ parallel to the magnetic field according to H.Fisser (13-moment approximation) are

$$\varkappa^{(i)} = \frac{25}{4}\,\frac{n}{\sigma}\,\frac{1}{\gamma^{(i)}}\,\frac{k}{m^{(i)}}\,kT^{(i)}$$

$$\sigma = \sqrt{2} + \frac{15}{2}\left(\frac{T^{(i)}}{T^{(e)}}\right)^{3/2}\left(\frac{m^{(e)}}{m^{(i)}}\right)^{1/2}$$

$$\gamma^{(i)} = \frac{4\sqrt{2\pi}}{3}\,\frac{e^4 n}{(m^{(i)})^{1/2}}\,\frac{1}{(kT^{(i)})^{3/2}}\,\ln\Lambda$$

$$\varkappa^{(e)} = \frac{25}{4}\,\frac{n}{\lambda}\,\frac{1}{\gamma^{(e)}}\,\frac{k}{m^{(e)}}\,kT^{(e)}$$

$$\lambda = \sqrt{2} + \frac{13}{4} = 4.6642$$

$$\gamma^{(e)} = \frac{4\sqrt{2\pi}}{3}\,\frac{e^4 n}{(m^{(e)})^{1/2}}\,\frac{1}{(kT^{(e)})^{3/2}}\,\ln\Lambda$$

$$\Lambda = \frac{3\,(kT^{(e)})^{3/2}}{2\,e^3\sqrt{\pi n}}$$

these formulae being essentially the same as Spitzer's /4/.

1.2 The system of equations

With due allowance for all of the above restrictions the fundamental equations of the model in question are as follows:

$$\frac{\partial}{\partial t}\,\rho(x_i,t) + \frac{\partial}{\partial x_\mu}\left(\rho(x_i,t)\,V_\mu(x_i,t)\right) = 0 \tag{1.1}$$

$$\rho(x_i,t)\left(\frac{\partial}{\partial t}V_\alpha(x_i,t) + V_\mu(x_i,t)\,\frac{\partial}{\partial x_\mu}V_\alpha(x_i,t)\right) + \frac{\partial}{\partial x_\alpha}P(x_i,t) - \epsilon_{\alpha ik}\,j_i\,B_k = 0 \tag{1.2}$$

$$\frac{\partial}{\partial t} P(x_i,t) + \frac{\partial}{\partial x_\mu}\Big(P(x_i,t)\, V_\mu(x_i,t)\Big) + \tag{1.3}$$
$$+ \frac{2}{3} P(x_i,t) \frac{\partial}{\partial x_\mu} V_\mu(x_i,t) - \frac{2}{3}\frac{\partial}{\partial x_\mu}\Big(\varkappa \frac{\partial}{\partial x_\mu} T(x_i,t)\Big) = 0$$

$$\frac{\partial}{\partial t}\vec{B}(\vec{x},t) = -\vec{\nabla}\times\vec{E}(\vec{x},t) \tag{1.4}$$

$$\vec{j}(\vec{x},t) = \vec{\nabla}\times\vec{B}(\vec{x},t) \tag{1.5}$$

$$\vec{\nabla}\cdot\vec{B}(\vec{x},t) = 0 \tag{1.6}$$

$$\vec{E} + \vec{v}\times\vec{B} = 0 \tag{1.7}$$

where special units have been chosen for the electromagnetic quantities, viz.

$$E \longrightarrow \frac{\sqrt{4\pi}}{c} E \qquad B \longrightarrow \sqrt{4\pi}\, B$$
$$j \longrightarrow \frac{c}{\sqrt{4\pi}} j \qquad \rho \longrightarrow \frac{4\pi}{c^2}\rho \tag{1.8}$$

(the conventional symbols on the LHS being replaced by the expressions on the RHS).

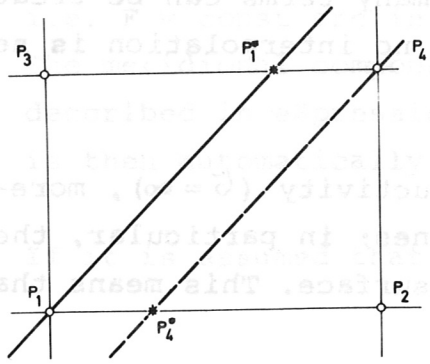
2. Coordinate system

The fundamental equations, a system of coupled partial differential equations for a region $\mathcal{G}$, are solved by using a coordinate system specially fitted to this particular problem.

On the one hand, we want as simple a coordinate system as possible (e.g. Cartesion or cylindrical coordinates) that conforms roughly to the geometry of the region. If, however, the region is not bounded by coordinate surfaces, this invariably leads to complicated boundary conditions (interpolation problems in numerical calculations). On the other hand, we can try to fit the coordinate system to the region (including the boundary), but this may only be possible with a complicated coordinate system and requires a more elaborate formulation of the system of equations (additional terms for transforming the equations).

Cylindrical coordinates seem to be the most promising way of numerically calculating the dynamics of a theta pinch discharge in view of its geometry. This, however, entails complicated interpolations for the boundaries. Two-dimensional programmes have hitherto used such orthogonal coordinates in r, z or non-orthogonal coordinates fitted to the coil shape /7/. In general, however, the magnetic field lines will not coincide with a set of coordinate lines. This means that many effects, particularly those occurring parallel to the magnetic field lines (e.g. thermal conduction), can be numerically calculated only with difficulty or else not exactly.

Example: Calculation of the thermal current in Cartesian coordinates in which the magnetic field lines are diagonal.



a) The thermal current from $P_1 \rightarrow P_1^*$ has to be distributed to P_3 and P_4.

b) Heat flows from $P_4^* \rightarrow P_4$. As P_4^* is not known, it has to be calculated from P_1, P_4 by interpolation.

Fig. 1

The errors resulting from this inaccuracy always lead to a numerical thermal current perpendicular to the field lines.

For this reason it is advisable to introduce a coordinate system in which a set of coordinate lines are magnetic field lines.

For the case of the theta pinch the surfaces r = const of the cylindrical coordinates are thus replaced by surfaces of constant magnetic flux.

2.1 Magnetic field lines as coordinate lines

The magnetic field lines in a theta pinch discharge do not, in general, lie in surfaces r = const. Furthermore, their position varies with time, firstly owing to the change of the coil field and secondly owing to the dynamics of the plasma.
Using the magnetic field lines as coordinate lines complicates the coordinate system in two ways:

1. It is curvilinear and non-orthogonal
2. It is itself a function of time.

Additional terms will thus be obtained when the fundamental equations are transformed to these coordinates.
A general restriction on the use of such a coordinate system is that there ought not to be any regions where B = 0. For theta pinch calculations this means that there has to be a magnetic field trapped in the plasma. (Presumably, however, it will be possible to connect regions without magnetic field to regions with magnetic field.)

But this is matched by the twofold advantage that pysical effects can be described more exactly, and that many terms can be treated more simply in these coordinates because no interpolation is necessary.

For the case of infinite electrical conductivity ($\sigma = \infty$), moreover, the plasma is tied to the field lines; in particular, the plasma-vacuum interface is a coordinate surface. This means that

the system of differential equations is simplified because the compressed plasma covers a fraction of the coil radius, and that numerical treatment by difference methods can be achieved with a reasonable size of computer (IBM 7090, approx. 32,000 storage locations).

2.11 Definition of the coordinate system

First it is assumed that the magnetic field has only meridional components B_r, B_z, i.e. the toroidal component B_φ vanishes. The flux function

$$F(r,z,t) := \int_0^r B_z(r',z,t)\, r'\, dr' \tag{2,1}$$

then describes the magnetic field by the field lines F(r,z,t)= const.

Differentiation with respect to r yields the B_z component, differentiation with respect to z yields the B_r component (because the magnetic field is solenoidal, $\vec{\nabla}\cdot\vec{B} = \frac{1}{r}\frac{\partial}{\partial r}(rB_r) + \frac{\partial}{\partial z}B_z = 0$):

$$B_z(r,z,t) = \frac{1}{r}\frac{\partial}{\partial r}F(r,z,t) \tag{2,2}$$

$$B_r(r,z,t) = -\frac{1}{r}\frac{\partial}{\partial z}F(r,z,t). \tag{2.3}$$

It can easily be checked that

$$\vec{B}\cdot\vec{\nabla}F \equiv 0$$

i.e. F = const are in fact field lines.

The meridional component of the magnetic field can readily be described in expressions of the flux function since $\vec{\nabla}\cdot\vec{B} = 0$ is then automatically satisfied.

If it is assumed that $B_z \neq 0$, the explicit representation of the

magnetic field lines is obtained by inverting the flux function:

$$r = R(F,z,t) \tag{2.4}$$

The three sets of coordinate surfaces are

$$F = \text{const}$$
$$\varphi = \text{const}$$
$$z = \text{const}.$$

The intersections of two coordinate surfaces yield the coordinate lines in each case.

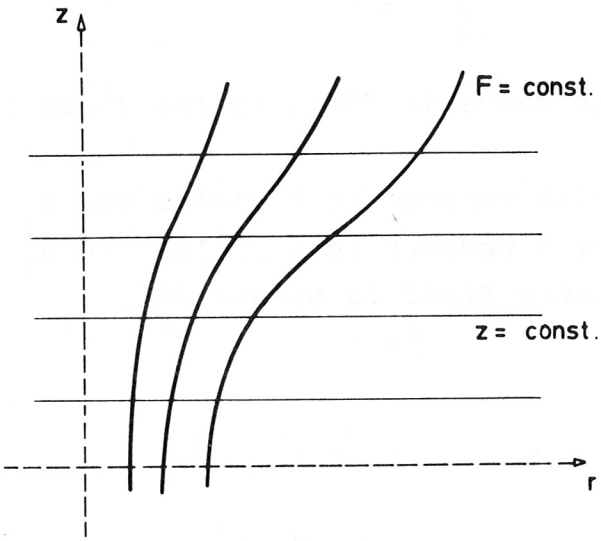


Fig. 2 Coordinate lines

Although the obvious course from the mathematical point of view is to use the orthogonal trajectories of F(r,z,t) instead of z = const and thus obtain a triply orthogonal coordinate system, this was rejected because these surfaces cannot be calculated directly in a similar way. Numerical calculation is not only time consuming, it also involves the risk of instability [8]. In order to derive the equation of motion for F(r,z,t), we consider the derivative of eq. (2.1) with respect to time:

$$\frac{\partial}{\partial t} F(r,z,t) = \int_0^r \frac{\partial}{\partial t} B_z(r',z,t)\, r' dr' \qquad (2.5)$$

Taking the z-component of the induction law (eq. (1.4))

$$\frac{\partial}{\partial t} B_z = -\frac{1}{r}\frac{\partial}{\partial r}(r E_\varphi)$$

and the φ-component of Ohm's law

$$E_\varphi = v_r B_z - v_z B_r + \eta\left(\frac{\partial}{\partial z} B_r - \frac{\partial}{\partial r} B_z\right),$$

where $j_\varphi = \frac{\partial}{\partial z} B_r - \frac{\partial}{\partial r} B_z$ is substituted for j_φ (eq. (1.5)),

we obtain from eq. (2.5)

$$\frac{\partial}{\partial t} F = -r E_\varphi$$

or

$$\frac{\partial}{\partial t} F = -r\left\{v_r B_z - v_z B_r + \eta\left(\frac{\partial}{\partial z} B_r - \frac{\partial}{\partial r} B_z\right)\right\} \qquad (2.6)$$

and, finally, with eqs. (2.2) and (2.3)

$$\frac{\partial F}{\partial t} + v_r \frac{\partial F}{\partial r} + v_z \frac{\partial F}{\partial z} = \eta\left[r \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial F}{\partial r}\right) + \frac{\partial^2 F}{\partial z^2}\right] \qquad (2.7)$$

which is the equation of motion for F(r,z,t).

For infinite electrical conductivity ($\eta = 0$) the substantial derivative is $\frac{D}{Dt}F = 0$, i.e. the plasma is tied to the field lines.

The equation of motion for R(F,z,t) is obtained by comparing the differentials (first for t = const) of F = F(r,z,t):

$$dF = \frac{\partial F}{\partial r} dr + \frac{\partial F}{\partial z} dz = r B_z dr - r B_r dz$$

r = R(F,z,t):

$$dr = dR = \frac{\partial R}{\partial F} dF + \frac{\partial R}{\partial z} dz$$

$$dF = \frac{1}{\left(\frac{\partial R}{\partial F}\right)} dR - \frac{\left(\frac{\partial R}{\partial z}\right)}{\left(\frac{\partial R}{\partial F}\right)} dz .$$

Comparison of the coefficients gives

$$r B_z = \frac{1}{\left(\frac{\partial R}{\partial F}\right)} \qquad R \frac{\partial R}{\partial F} = \frac{1}{B_z} \tag{2.8}$$

$$r B_r = \frac{\left(\frac{\partial R}{\partial z}\right)}{\left(\frac{\partial R}{\partial F}\right)} \qquad \frac{\partial R}{\partial z} = \frac{B_r}{B_z} . \tag{2.9}$$

Comparison of the time dependent differentials

$$dF = \frac{\partial F}{\partial r} dr + \frac{\partial F}{\partial z} dz + \frac{\partial F}{\partial t} dt$$

and

$$dR = \frac{\partial R}{\partial F} dF + \frac{\partial R}{\partial z} dz + \frac{\partial R}{\partial t} dt$$

yields $\frac{\partial R}{\partial t}$.

For this purpose we compare the positions of a point with given z on the the same field line F for t and t + dt, i.e. dF = dz = 0. We then get

$$\frac{\partial R}{\partial t} = - \frac{\left(\frac{\partial F}{\partial t}\right)}{\left(\frac{\partial F}{\partial r}\right)} . \qquad (2.10)$$

With eq. (2.6) and $\frac{\partial F}{\partial r} = - \frac{1}{\left(\frac{\partial R}{\partial F}\right)}$

we finally obtain

$$\frac{\partial R}{\partial t} = v_r + v_z \frac{\left(\frac{\partial F}{\partial z}\right)}{\left(\frac{\partial F}{\partial r}\right)} - \eta \left[r \frac{\partial}{\partial r} \left(\frac{1}{r} \frac{\partial F}{\partial r}\right) + \frac{\partial^2 F}{\partial z^2} \right] \Big/ \left(\frac{\partial F}{\partial r}\right) \qquad (2.11)$$

the equation of motion for R(F,z,t).

2.12 B_φ -component of the magnetic field

The flux function F(r,z,t) and the new coordinate lines R(F,z,t) were introduced in case B_φ vanish. It shall now be shown that the remarks in 2.11 are also applicable to the case $B_\varphi \neq 0$.

The induction law in the case of azimuthal symmetry ($\frac{\partial}{\partial \varphi} = 0$) is (in cylindrical coordinates)

$$\frac{\partial}{\partial t} B_r = \frac{\partial}{\partial z} E_\varphi$$

$$\frac{\partial}{\partial t} B_\varphi = - \left(\frac{\partial}{\partial z} E_r - \frac{\partial}{\partial r} E_z\right)$$

$$\frac{\partial}{\partial t} B_z = - \frac{1}{r} \frac{\partial}{\partial r} (r E_\varphi).$$

The electric field $\vec{E} = \left\{ E_r, E_\varphi, E_z \right\}$ is governed by Ohm's Law:

$$\vec{E} = - \vec{v} \times \vec{B} + \eta \vec{\nabla} \times \vec{B} - \frac{1}{e\,n} \left[\vec{\nabla} P^{(e)} - (\vec{\nabla} \times \vec{B}) \times \vec{B} \right].$$

The flux function F(r,z,t) can now be defined as in 2.11. As the condition $\vec{\nabla} \times \vec{B} = 0$ is not changed by $B_\varphi \neq 0$, we can calculate B_r and B_z in the same way.
For the equation of motion of F(r,z,t) eq. (2.6)

$$\frac{\partial F}{\partial t} = - r E_\varphi$$

is also valid, but E_φ now depends on B_φ by way of the Hall term. This means that for $B_\varphi \neq 0$ the field lines are no longer in meridional planes φ = const, but nevertheless in the surfaces F = const, i.e. there are no components of $\vec{B}$ perpendicular to F = const.

2.13 Vacuum magnetic field

Because of eq. (1.5) it holds for a vacuum field where $\vec{j} \equiv 0$ that

$$\vec{\nabla} \times \vec{B} = 0$$

which is expressed in components as follows:

$$\frac{\partial}{\partial z} B_\varphi = 0 \qquad (2.12)$$

$$\frac{\partial}{\partial z} B_r - \frac{\partial}{\partial r} B_z = 0$$

$$\frac{\partial}{\partial r} (r B_\varphi) = 0 .$$

With eqs. (2.2) and (2.3) we get for the φ-component

$$\frac{\partial^2 F}{\partial r^2} - \frac{1}{r} \frac{\partial F}{\partial r} + \frac{\partial^2 F}{\partial z^2} = 0. \qquad (2.13)$$

This is a second order partial differential equation (of the elliptic type) known as Stoke's differential equation (same as the Laplace equation except for the minus sign of the second term).

The same type of differential equation for $R(F,z)$ in the F-z space is obtained in the following way:

We consider B_r and B_z as functions of $F(r,z)$ and z thus:

$$B_i(r,z) = B_i(R(F,z),z) = \tilde{B}_i(F,z) = \tilde{B}_i(F(r,z),z)$$

$$\frac{\partial B_r}{\partial z} = \frac{\partial \tilde{B}_r}{\partial F}\frac{\partial F}{\partial z} + \frac{\partial \tilde{B}_r}{\partial z}$$

$$\frac{\partial B_z}{\partial r} = \frac{\partial \tilde{B}_z}{\partial F}\frac{\partial F}{\partial r} .$$

Equation (2.12) $\frac{\partial}{\partial z} B_r - \frac{\partial}{\partial r} B_z = 0$ is then written first in the form

$$\frac{\partial \tilde{B}_r}{\partial F}\frac{\partial F}{\partial z} + \frac{\partial \tilde{B}_r}{\partial z} - \frac{\partial \tilde{B}_z}{\partial F}\frac{\partial F}{\partial r} = 0 .$$

With eqs. (2.2) and (2.3) this becomes

$$\frac{\partial \tilde{B}_r}{\partial F}(- r\,\tilde{B}_r) + \frac{\partial \tilde{B}_r}{\partial z} - \frac{\partial \tilde{B}_z}{\partial F}(r\,\tilde{B}_z) = 0 . \qquad (2.14)$$

From eqs. (2.8) and (2.9) we get

$$\tilde{B}_r = \frac{R_z}{R\,R_F} \qquad (2.15)$$

$$\tilde{B}_z = \frac{1}{R\,R_F} . \qquad (2.16)$$

Substituting the relations (2.15) and (2.16) for B_r and B_z in eq. (2.14) yields

$$\left(- \frac{R_z}{R_F}\frac{\partial}{\partial F} + \frac{\partial}{\partial z} \right)\left(\frac{R_z}{R\,R_F} \right) - \frac{1}{R_F}\frac{\partial}{\partial F}\left(\frac{1}{R\,R_F} \right) = 0$$

which is differentiated to give a differential equation for the field lines $R = R(F,z)$

$$(1 + R_z^{\,2})R_{FF} - 2\, R_F\, R_z\, R_{Fz} + R_F^{\,2}\, R_{zz} + \frac{1}{R}\, R_F^{\,2} = 0. \qquad (2.17)$$

Near the axis ($R \rightarrow 0$) all z derivatives vanish. For this limiting case eq. (2.17) becomes

$$R_{FF} + \frac{R_F^{\,2}}{R} = 0 = \frac{\partial}{\partial F}(R\, R_F) .$$

After integration we get first

$$R\, R_F = \frac{1}{2}\, \mathrm{const}^2$$

and finally

$$\frac{1}{2}\, R^2 = \frac{1}{2}\, \mathrm{const}^2 \cdot F + \mathrm{const}' .$$

For $R = 0$ we also get $F = 0$, i.e. $\mathrm{const}' = 0$. That is, for $R \rightarrow 0$ the solution of the differential equation (2.17) is

$$R = \mathrm{const} \cdot F^{1/2} .$$

Since working with the square root presents numerical problems, a new variable s is introduced thus:

$$F = \frac{s^2}{2} . \qquad (2.18)$$

The differential equation for the field lines $R(F,z)$ (eq. (2.17)) then becomes a differential equation for $R(s,z)$

$$(1 + R_z^2)\,(R_{ss} - \frac{1}{s} R_s) - 2\, R_s R_z R_{sz} + R_s^2\, R_{zz} + \frac{R_s^2}{R} . \qquad (2.19)$$

The relation between the field lines and the magnetic field is now

$$B_z = \frac{s}{R\, R_s} \qquad (2.20)$$

$$B_r = \frac{s}{R\, R_s} R_z . \qquad (2.21)$$

For further discussions the following operators are defined:

$$L_c\,[F] \equiv r \frac{\partial}{\partial r} (\frac{1}{r} \frac{\partial F}{\partial r}) + \frac{\partial^2 F}{\partial z^2} \qquad (2.22)$$

$$L_F\,[R] \equiv (1 + R_z^2)\, R_{FF} - 2\, R_F R_z R_{Fz} + R_F^2\, R_{zz} + \frac{R_F^2}{R} \qquad (2.23)$$

$$L_S\,[R] \equiv (1 + R_z^2)\,(R_{ss} - \frac{1}{s} R_s) - 2\, R_s R_z R_{sz} + R_s^2\, R_{zz} + \frac{R_s^2}{R} . \qquad (2.24)$$

The physical component of the current in the azimuthal direction is

$$\hat{j}_\varphi = -\frac{1}{r} L_c\,[F] = \frac{s}{R\, R_s^3} L_s\,[R] . \qquad (2.25)$$

2.2 General transformation formulae

2.21 Coordinate transformation

The formulae for transformation from Cartesian coordinates $\{x^1, x^2, x^3\}$ (abbreviated in the following to KS1) to general (curvilinear, non-orthogonal, time dependent) coordinates $\{\xi^1, \xi^2, \xi^3\}$ (abbreviated to KS2) are written in general representation thus:

$$x^i = x^i(\xi^1, \xi^2, \xi^3, t). \tag{2.26}$$

The differential form of this is

$$dx^i = \frac{\partial x^i}{\partial \xi^k} d\xi^k + \frac{\partial x^i}{\partial t} dt, \tag{2.27}$$

where the $\frac{\partial x^i}{\partial \xi^k}$ are the transformation coefficients of a contravariant vector and $\frac{\partial x^i}{\partial t}$ the velocity of the new coordinate system (KS2) in the old (KS1).

The transformation equations in our special case are

$$\{x^i\} \longrightarrow \boxed{\begin{aligned} x^1 &= R(s,z,t)\cos\varphi \\ x^2 &= R(s,z,t)\sin\varphi \\ x^3 &= z \end{aligned}} \longrightarrow \{\xi^k\} \equiv \{s, \varphi, z\}. \tag{2.28}$$

The Christoffel symbols of the second kind are calculated to give

$$\Gamma^1_{11} = \frac{R_{ss}}{R_s} \qquad \Gamma^1_{22} = -\frac{R}{R_s} \qquad \Gamma^1_{33} = \frac{R_{zz}}{R_s}$$

$$\Gamma^2_{12} = \Gamma^2_{21} = \frac{R_s}{R} \qquad \Gamma^1_{13} = \Gamma^1_{31} = \frac{R_{sz}}{R_s} \qquad \Gamma^2_{23} \quad \Gamma^2_{32} = \frac{R_z}{R}$$

All other Γ^i_{jk} vanish.

The velocity of the new coordinate system $\{\xi^k\}$ (KS2) relative to the old $\{x^i\}$ (KS1) is obtained from eq. (2.27). We consider a point $d\xi^k = 0$ fixed in KS2. After a time dt it undergoes in KS1 the shift

$$dx^i = \frac{\partial x^i}{\partial t}\, dt\,.$$

We thus get

$$w^i(x,t) = \frac{\partial x^i}{\partial t} = \begin{pmatrix} \frac{\partial R}{\partial t}\cos\varphi \\ \frac{\partial R}{\partial t}\sin\varphi \\ 0 \end{pmatrix} = \bar{w}^i(\xi,t) \qquad (2.29)$$

as the velocity of the new system $\{\xi^k\}$ relative to the old $\{x^i\}$.

The velocity $\bar{\bar{w}}^k$ of KS1 in KS2 is obtained in the same way from eq. (2.27) with $dx^i = 0$.

$$0 = \frac{\partial x^i}{\partial \xi^k}\, d\xi^k + \frac{\partial x^i}{\partial t}\, dt\,.$$

Multiplying by $\frac{\partial \xi^\nu}{\partial x^i}$ and dividing by dt gives

$$0 = \frac{\partial x^i}{\partial \xi^k}\frac{\partial \xi^\nu}{\partial x^i}\left(\frac{d\xi^k}{dt}\right)_x + \frac{\partial \xi^\nu}{\partial x^i}\frac{\partial x^i}{\partial t}\,.$$

Because of

$$\left(\frac{d\xi^k}{dt}\right)_x = \frac{\partial \xi^k}{\partial t} \equiv \bar{\bar{w}}^k$$

we get

$$0 = \delta_k^{\ \nu}\,\bar{\bar{w}}^k + \frac{\partial \xi^\nu}{\partial x^i}\frac{\partial x^i}{\partial t}$$

and hence

$$\bar{\bar{w}}^\nu = -\frac{\partial \xi^\nu}{\partial x^i}\frac{\partial x^i}{\partial t} = -\frac{\partial \xi^\nu}{\partial x^i}\,\bar{w}^i(\xi,t) = \begin{pmatrix} -\frac{1}{R_s}\frac{\partial}{\partial t}R \\ 0 \\ 0 \end{pmatrix} = \bar{\bar{w}}^\nu(\xi,t). \qquad (2.30)$$

We now transform the various quantities.

The mass density ρ (scalar)

$$\rho(x,t) = \rho(\, x(\xi,t),t\,) = \bar{\rho}(\,\xi(x,t),t\,) = \tilde{\rho}(\xi,t)$$

is invariant under transformations (KS1 $\longrightarrow$ KS2)

$$\bar{\rho}(\xi,t) \equiv \tilde{\rho}(\xi,t). \tag{2.31}$$

Contravariant vectors (contravariant tensors of rank 1) transform as

$$\bar{A}^i(\xi,t) = \frac{\partial x^i}{\partial \xi_\alpha} \tilde{A}^\alpha(\xi,t). \tag{2.32}$$

The velocity of KS2 $\{\xi^k\}$ in KS1 $\{x^i\}$ is thus

$$\bar{w}^i(\xi,t) = \frac{\partial x^i}{\partial \xi_\alpha} \tilde{w}^\alpha(\xi,t)$$

or

$$\tilde{w}^\alpha(\xi,t) = \frac{\partial \xi^\alpha}{\partial x^i} \bar{w}^i(\ ,t) = \begin{pmatrix} \frac{1}{R_s}\frac{\partial}{\partial t} R \\ 0 \\ 0 \end{pmatrix} = -\bar{\bar{w}}^\alpha. \tag{2.33}$$

The contravariant vector components of the velocity

$v^i(x,t) = \bar{v}^i(\xi,t)$ thus transform as

$$\bar{v}^i(\xi,t) = \frac{\partial x^i}{\partial \xi_\alpha} \tilde{v}^\alpha(\xi,t) + \bar{w}^i$$

$$\bar{v}^i(\xi,t) = \frac{\partial x^i}{\partial \xi_\alpha} (\,\tilde{v}^\alpha + \tilde{w}^\alpha\,)\,. \tag{2.34}$$

The transformation of the contravariant components of the pressure tensor $P^{ik}(x,t) = P^{ik}(\, x(\xi,t),t\,) = \bar{P}^{ik}(\xi,t)$ is

$$\bar{P}^{ik} = \frac{\partial x^i}{\partial \xi_\alpha}\frac{\partial x^k}{\partial \xi_\beta}\,\tilde{P}^{\alpha\beta}(\xi,t) \tag{2.35}$$

or conversely

$$\tilde{P}^{\alpha\beta}(\xi,t) = \frac{\partial \xi^\alpha}{\partial x^i}\frac{\partial \xi^\beta}{\partial x^k}\,\bar{P}^{ik} .$$

In the new coordinate system KS2 a scalar pressure $P^{ik} = p\,\delta^{ik}$ will thus be of the form

$$\tilde{P}^{\alpha\beta} = g^{\alpha\beta}\bar{P} = \begin{pmatrix} \frac{1+R_z^{\,2}}{R_s^{\,2}} & 0 & -\frac{R_z}{R_s} \\ 0 & \frac{1}{R^2} & 0 \\ -\frac{R_z}{R_s} & 0 & 1 \end{pmatrix} \bar{P}. \tag{2.36}$$

The pressure tensor transforms like a contravariant tensor owing to the definition of the pressure

$$P^{\alpha\beta}(x^i,t) = m\int_{-\infty}^{+\infty} u^\alpha\, u^\beta\, f(x^i,w^k,t)\; d^3w_k ,$$

where u^μ is the peculiar velocity (difference between the particle velocity and mean mass velocity v^μ). Thus, the velocity terms of the coordinate system that appear when the velocity components are transformed cancel. For the sake of clarity the transformations of the various types of differential expressions that occur in the fundamental equations are set out below.

The transformation of the derivative with respect to time $\frac{\partial}{\partial t} A^{ik}$ is as follows:

$$\frac{\partial}{\partial t} A^{ik}(x,t)\Big/_{x=\text{const}} = \frac{\partial}{\partial t}\left(\frac{\partial x^i}{\partial \xi_\alpha}\frac{\partial x^k}{\partial \xi_\beta}\tilde{A}^{\alpha\beta}\right)\Big/_{\xi} - \tilde{w}^\nu \frac{\partial}{\partial \xi_\nu}\left(\frac{\partial x^i}{\partial \xi_\alpha}\frac{\partial x^k}{\partial \xi_\beta}\tilde{A}^{\alpha\beta}\right)\Big/_{t} . \tag{2.37}$$

The terms of the form $v^\mu \frac{\partial}{\partial x_\mu} A^{ik}$ transform as follows:

$$v^\mu \frac{\partial}{\partial x_\mu} A^{ik}(x,t) = (\tilde{v}^\nu + \tilde{w}^\nu)\frac{\partial}{\partial \xi_\nu}\left[\frac{\partial x^i}{\partial \xi_\alpha}\frac{\partial x^k}{\partial \xi_\beta}\tilde{A}^{\alpha\beta}(\xi,t)\right] . \tag{2.38}$$

For the transformation of the divergences $\frac{\partial}{\partial x_\mu} A^{i\mu}(x,t)$ we get

$$\frac{\partial}{\partial x_\mu} A^{i\mu}(x,t) = \frac{\partial x^i}{\partial \xi_\alpha}\left[\frac{\partial}{\partial \xi_\nu}\tilde{A}^{\alpha\nu} + \tilde{A}^{\alpha\mu}\Gamma^{\nu}_{\mu\nu} + \tilde{A}^{\mu\nu}\Gamma^{\alpha}_{\mu\nu}\right] = \frac{\partial x^i}{\partial \xi_\alpha}(\tilde{A}^{\alpha\nu})_{,\nu} . \tag{2.39}$$

The expression $\frac{\partial}{\partial \xi_\nu}\left[\frac{\partial x^i}{\partial \xi_\alpha}\frac{\partial x^k}{\partial \xi_\beta}\tilde{A}^{\alpha\beta}\right]$ in eq. (2.38) can be rewritten in the form

$$\frac{\partial}{\partial \xi_\nu}\left[\frac{\partial x^i}{\partial \xi_\alpha}\frac{\partial x^k}{\partial \xi_\beta}\tilde{A}^{\alpha\beta}\right] = \frac{\partial x^i}{\partial \xi_\alpha}\frac{\partial x^k}{\partial \xi_\beta}(\tilde{A}^{\alpha\beta})_{,\nu} \tag{2.40}$$

where

$$(\tilde{A}^{\alpha\beta})_{,\nu} = \frac{\partial}{\partial \xi_\nu}\tilde{A}^{\alpha\beta} + \tilde{A}^{\alpha\mu}\Gamma^{\beta}_{\mu\nu} + \tilde{A}^{\mu\beta}\Gamma^{\alpha}_{\mu\nu} . \tag{2.41}$$

The expression $\frac{\partial}{\partial t}\left[\frac{\partial x^i}{\partial \xi_\alpha}\frac{\partial x^k}{\partial \xi_\beta}\tilde{A}^{\alpha\beta}\right]$ in eq. (2.37) can be further rewritten to give

$$\begin{aligned}\frac{\partial}{\partial t}\left[\frac{\partial x^i}{\partial \xi_\alpha}\frac{\partial x^k}{\partial \xi_\beta}\tilde{A}^{\alpha\beta}\right] &= \frac{\partial x^i}{\partial \xi_\alpha}\frac{\partial x^k}{\partial \xi_\beta}\frac{\partial}{\partial t}\tilde{A}^{\alpha\beta} + \\ &+ \tilde{A}^{\alpha\beta}\left\{\frac{\partial x^i}{\partial \xi_\alpha}\frac{\partial x^k}{\partial \xi_\nu}(\tilde{w}^\nu)_{,\beta} + \frac{\partial x^i}{\partial \xi_\nu}\frac{\partial x^k}{\partial \xi_\beta}(\tilde{w}^\nu)_{,\alpha}\right\} .\end{aligned} \tag{2.42}$$

2.22 Transformation of the individual equations

Transformation of the continuity equation

The individual terms of the continuity equation (eq. (1.1)) transform as shown in 2.21. The continuity equation in the new coordinates is now written

$$\frac{\partial}{\partial t}\tilde{\rho} + \frac{\partial}{\partial \xi_\mu}(\tilde{\rho}\tilde{v}^\mu) + \tilde{\rho}\,\tilde{v}^\nu \Gamma^\mu_{\nu\mu} + \tilde{\rho}\left(\frac{\partial}{\partial \xi_\mu}\tilde{w}^\mu + \tilde{w}^\nu \Gamma^\mu_{\nu\mu}\right) = 0. \quad (2.43)$$

Here the first two terms are formally equal to the corresponding terms of the continuity equation in Cartesian coordinates. In addition, because of the non-orthogonality of the coordinates we get the term $\tilde{v}^\nu \Gamma^\mu_{\nu\mu}$ (which vanishes as soon as Cartesian coordinates are adopted).

The last term $\tilde{\rho}(\tilde{w}^\mu)_{,\mu}$, the divergence of the velocity of the field lines, allows for the time dependence of the coordinates, i.e. compression due to the change of coordinate system.

Transformation of the equation of motion

The equation of motion (eq. (1.2)) in KS1 is

$$\rho(x,t)\left(\frac{\partial}{\partial t} v^i(x,t) + v^\mu \frac{\partial}{\partial x_\mu} v^i\right) + F^i(x,t) = 0$$

where

$$F^i(x,t) = F^i_{(p)} + F^i_{(B)}$$

$$F^i_{(p)} = \frac{\partial}{\partial x_\mu} P^{i\mu}(x,t)$$

$$F^i_{(B)} = -\epsilon_{ikl}\, j_k\, B_l\,.$$

Transformation to KS2 in accordance with eqs. (2.37), (2.38), (2.39) yields

$$\frac{\partial}{\partial t}\left(\frac{\partial x^i}{\partial \xi_\alpha}(\tilde{v}^\alpha + \tilde{w}^\alpha)\right) - \tilde{w}^\nu \frac{\partial}{\partial \xi_\nu}\left(\frac{\partial x^i}{\partial \xi_\alpha}(\tilde{v}^\alpha + \tilde{w}^\alpha)\right) +$$
$$+ (\tilde{v}^\nu + \tilde{w}^\nu)\frac{\partial}{\partial \xi_\nu}\left(\frac{\partial x^i}{\partial \xi_\alpha}(\tilde{v}^\alpha + \tilde{w}^\alpha)\right) + \frac{\partial x^i}{\partial \xi_\alpha}\frac{\tilde{F}^\alpha}{\tilde{\rho}} = 0\,.$$

The second term again cancels (as in the continuity equation) with part of the third term. The first term and the rest of the third term are rewritten according to eq. (2.40) or (2.41) to give

$$\frac{\partial x^i}{\partial \xi_\alpha}\left\{\frac{\partial}{\partial t}(\tilde{v}^\alpha + \tilde{w}^\alpha) + (\tilde{v}^\nu + \tilde{w}^\nu)(\tilde{w}^\alpha)_{,\nu} + \tilde{v}^\nu(\tilde{v}^\alpha + \tilde{w}^\alpha)_{,\nu} + \frac{1}{\rho}\tilde{F}^\alpha\right\} = 0$$

which finally is rearranged as

$$\frac{\partial}{\partial t}\tilde{w}^\alpha + 2\,\tilde{v}^\mu(\tilde{w}^\alpha)_{,\mu} + \tilde{w}^\mu(\tilde{w}^\alpha)_{,\mu} + \frac{\partial}{\partial t}\tilde{v}^\alpha + \tilde{v}^\mu(\tilde{v}^\alpha)_{,\mu} + \frac{1}{\rho}\tilde{F}^\alpha = 0 \,. \tag{2.44}$$

The forces transform as follows:
The pressure force $F^i_{(p)}$ according to eq. (2.39) is

$$F^i_{(p)}(x,t) = \frac{\partial x^i}{\partial \xi_\alpha} F^\alpha_{(p)}(\xi,t) = \frac{\partial x^i}{\partial \xi_\alpha}(\tilde{p}^{\alpha\nu})_{,\nu} \,.$$

For the scalar pressure $P^{ik}(x,t) = \bar{P}^{ik}(\xi,t) = \bar{p}\,\delta_{ik}$ we get from eq. (2.36).

$$\tilde{p}^{\alpha\nu}(\xi,t) = g^{\alpha\nu}\bar{p} \,.$$

Because of $(g^{ik})_{,k} = 0$ this then gives

$$F^\alpha_{(p)}(\xi,t) = \bar{p}\,(g^{\alpha\nu})_{,\nu} + g^{\alpha\nu}(\bar{p})_{,\nu} = g^{\alpha\nu}\frac{\partial}{\partial \xi_\nu}\bar{p} \,. \tag{2.45}$$

To transform the Lorentz force $F^i_{(B)}$ we first rewrite in Cartesian coordinates (KS1) as follows. We regard $\vec{j}$ as a contravariant vector and write $F^i_{(B)}$ as

$$F^i_{(B)} = -B^i_k\, j^k \,,$$

where $B^i{}_k$ is then a mixed tensor of rank 2 the components of which are given by the coordinates (B_1, B_2, B_3) of the vector $\vec{B}(x,t)$:

$$B^i{}_k = \begin{array}{c} \\ \\ \\ \end{array}\begin{array}{cc} & \xrightarrow{\quad k \quad} \\ \downarrow i & \begin{pmatrix} 0 & B_3 & -B_2 \\ -B_3 & 0 & B_1 \\ B_2 & -B_1 & 0 \end{pmatrix} \end{array} . \tag{2.46}$$

The current $\vec{j}$ is given according to eq. (1.5) by $\vec{j} = \vec{\nabla} \times \vec{B}$. This can by written

$$j^m = \frac{\partial}{\partial x_n} B^{mn} ,$$

where the purely contravariant tensor B^{mn} is in turn given by

$$B^{mn} = \begin{array}{cc} & \xrightarrow{\quad n \quad} \\ \downarrow m & \begin{pmatrix} 0 & B_3 & -B_2 \\ -B_3 & 0 & B_1 \\ B_2 & -B_1 & 0 \end{pmatrix} \end{array} . \tag{2.47}$$

The Lorentz force is thus

$$F^i_{(B)}(x,t) = -B^i{}_k \, j^k = -B^i{}_k \frac{\partial}{\partial x_\ell} B^{k\ell} . \tag{2.48}$$

This expression is now transformed from KS1 to KS2 according to

$$B^i{}_k = \frac{\partial x^i}{\partial \xi_\alpha} \tilde{B}^\alpha{}_\beta \frac{\partial \xi^\beta}{\partial x_k}$$

$$B^{k\ell} = \frac{\partial x^k}{\partial \xi_\alpha} \frac{\partial x^\ell}{\partial \xi_\nu} \tilde{B}^{\alpha\nu}$$

$$j^k = \frac{\partial}{\partial x_\ell} B^{k\ell} = \frac{\partial x^k}{\partial \xi_\gamma} (\tilde{B}^{\gamma\nu})_{,\nu} = \frac{\partial x^k}{\partial \xi_\gamma} \tilde{j}^\gamma .$$

Consequently,

$$F^i_{(B)}(x,t) = - B^i{}_k \frac{\partial}{\partial x_\ell} B^{k\ell} = - \frac{\partial x^i}{\partial \xi_\alpha} \tilde{B}^\alpha{}_\beta \frac{\partial \xi^\beta}{\partial x_k} \frac{\partial x^k}{\partial \xi_\gamma} (\tilde{B}^{\gamma\nu})_{,\nu}$$

$$= - \frac{\partial x^i}{\partial \xi_\alpha} \left[\tilde{B}^\alpha{}_\beta (\tilde{B}^{\beta\nu})_{,\nu} \right] = + \frac{\partial x^i}{\partial \xi_\alpha} \tilde{F}^\alpha_{(B)}(\xi,t)$$

and so

$$\tilde{F}^{\alpha}_{(B)}(\xi,t) = -\tilde{B}^{\alpha}_{\beta}\,(\tilde{B}^{\beta\nu})_{,\nu} = -\tilde{B}^{\alpha}_{\beta}\,\tilde{j}^{\beta} \qquad (2.49)$$

where

$$\tilde{B}^{\alpha}_{\beta} = \frac{\partial\xi^{\alpha}}{\partial x^{i}}\,B^{i}{}_{k}\,\frac{\partial x^{k}}{\partial\xi_{\beta}} \qquad (2.50)$$

$$\tilde{B}^{\alpha\beta} = \frac{\partial\xi^{\alpha}}{\partial x^{i}}\,\frac{\partial\xi^{\beta}}{\partial x^{\ell}}\,B^{k\ell} \qquad (2.51)$$

$$\tilde{j}^{\beta} = (\tilde{B}^{\beta\nu})_{,\nu} = \frac{\partial}{\partial\xi_{\nu}}\,\tilde{B}^{\beta\nu} + \tilde{B}^{\mu\nu}\Gamma^{\beta}_{\mu\nu} + \tilde{B}^{\beta\mu}\Gamma^{\nu}_{\mu\nu}\,. \qquad (2.52)$$

Transformation of the pressure equation

The equation for isotropic pressure eq. (1.3) transforms analogously to the continuity equation

$$\frac{\partial}{\partial t}\,\bar{p} + \tilde{v}^{\nu}\,\frac{\partial}{\partial\xi_{\nu}}\,\bar{p} + \frac{5}{3}\,\bar{p}\,(\tilde{v}^{\nu} + \tilde{w}^{\nu})_{,\nu} + \frac{2}{3}\,(\tilde{S}^{\mu})_{,\nu} = 0$$

or

$$\frac{\partial}{\partial t}\,\bar{p} + \frac{\partial}{\partial\xi_{\nu}}\,(\tilde{v}^{\nu}\,\bar{p}) + \frac{2}{3}\,\bar{p}\,\frac{\partial}{\partial\xi_{\nu}}\,\tilde{v}^{\nu} + \frac{5}{3}\,\bar{p}\left[(\tilde{v}^{\nu} + \tilde{w}^{\nu})\Gamma^{\mu}_{\mu\nu} + \frac{\partial}{\partial\xi_{\nu}}\,\tilde{w}^{\nu}\right] + \frac{2}{3}\left[\frac{\partial}{\partial\xi_{\mu}}\,\tilde{S}^{\mu} + \tilde{S}^{\nu}\Gamma^{\mu}_{\mu\nu}\right] = 0\,. \qquad (2.53)$$

2.3 Fundamental equations in rotational symmetric magnetic field coordinates

The last step to the final explicit form of the fundamental equations in magnetic field coordinates is merely to insert the Christoffel symbols in eqs. (2.43, (2.44), and (2.53). This yields the following expressions, which are abbreviated as shown:

$$\Gamma^{\nu}_{\nu 1} = \frac{R_{ss}}{R_s} + \frac{R_s}{R} = \frac{\partial}{\partial s}\ln(RR_s) =: GS \qquad (2.54)$$

$$\Gamma^{\nu}_{\nu 3} = \frac{R_{sz}}{R_s} + \frac{R_z}{R} = \frac{\partial}{\partial z}\ln(RR_s) =: GZ \qquad (2.55)$$

$$(\tilde{w}^{\nu})_{,\nu} = \frac{\partial}{\partial \xi_{\nu}} \tilde{w}^{\nu} + \tilde{w}^{\mu} \Gamma^{\nu}_{\nu\mu} = \frac{\partial}{\partial s}\left(\frac{R_t}{R_s}\right) + \left(\frac{R_t}{R_s}\right)\left(\frac{R_{ss}}{R_s} + \frac{R_s}{R}\right) =$$

$$= \frac{R_{st}}{R_s} + \frac{R_t}{R} = \frac{\partial}{\partial t} \ln (RR_s) =: \text{DIV } W . \tag{2.56}$$

2.31 Continuity equation

The continuity equation is then

$$\frac{\partial}{\partial t} \tilde{\rho} + \frac{\partial}{\partial s} (\tilde{\rho}\, \tilde{v}^s) + \frac{\partial}{\partial z} (\tilde{\rho}\, v^z) + \tilde{\rho} \left[\tilde{v}^s \text{ GS} + \tilde{v}^z \text{ GZ} + \text{DIV } W \right] = 0$$

or in another form (2.57)

$$\frac{\partial}{\partial t} D + \frac{\partial}{\partial s} (D\, \tilde{v}^s) + \frac{\partial}{\partial z} (D\, \tilde{v}^z) = 0 \tag{2.58}$$

with $D := \tilde{\rho}\,(RR_s)$.

2.32 Equation of motion

For the equation of motion (2.44) we get

$$\begin{pmatrix} \frac{\partial}{\partial t} \tilde{w}^s \\ 0 \\ 0 \end{pmatrix} + 2 \begin{pmatrix} \left(\tilde{v}^s \frac{\partial}{\partial s} + \tilde{v}^z \frac{\partial}{\partial z}\right) \tilde{w}^s + \tilde{w}^s \frac{R_{ss}}{R_s} + \tilde{v}^z \frac{R_{sz}}{R_s} \\ \tilde{w}^s \left(\tilde{v}^{\varphi} \frac{R_s}{R}\right) \\ 0 \end{pmatrix} + \begin{pmatrix} \tilde{w}^s \left(\frac{\partial}{\partial s} \tilde{w}^s + \tilde{w}^s \frac{R_{ss}}{R_s}\right) \\ 0 \\ 0 \end{pmatrix} + \tag{2.59}$$

$$+ \left(\frac{\partial}{\partial t} + \tilde{v}^s \frac{\partial}{\partial s} + \tilde{v} \frac{\partial}{\partial z}\right) \begin{pmatrix} \tilde{v}^s \\ \tilde{v}^{\varphi} \\ \tilde{v}^z \end{pmatrix} + \begin{pmatrix} \frac{1}{R_s} \left\{ \tilde{v}^s (R_{ss} \tilde{v}^s + 2 R_{sz} \tilde{v}^z) - R \tilde{v}^{\varphi} \tilde{v}^{\varphi} + R_{zz} \tilde{v}^z \tilde{v}^z \right\} \\ \frac{1}{R}\, 2 \tilde{v}^{\varphi} (R_s \tilde{v}^s + R_z \tilde{v}^z) \\ 0 \end{pmatrix} +$$

$$+ \frac{1}{\tilde{\rho}} \begin{pmatrix} \tilde{F}^s \\ \tilde{F}^{\varphi} \\ \tilde{F}^z \end{pmatrix} = 0$$

With $\tilde{F}^{\alpha} = \tilde{F}^{\alpha}_{(p)} + \tilde{F}^{\alpha}_{(B)}$.

We find for $F^{\alpha}_{(p)}$ (according to eq. (2.45))

$$\tilde{F}^{\alpha}_{(p)} = \begin{pmatrix} \left(\frac{1 + R_z^{\,2}}{R_s^{\,2}} \frac{\partial}{\partial s} - \frac{R_z}{R_s} \frac{\partial}{\partial z} \right) \bar{p} \\ 0 \\ \left(- \frac{R_z}{R_s} \frac{\partial}{\partial s} + \frac{\partial}{\partial z} \right) \bar{p} \end{pmatrix} . \qquad (2.60)$$

For the explicit expression of $\tilde{F}^{\alpha}_{(B)}$ (eq. (2.49)) we first have to calculate $\tilde{B}^{\alpha}_{\beta}(\xi,t)$, $\tilde{B}^{\alpha\beta}(\xi,t)$, and $\tilde{j}^{\beta}(\xi,t)$. In accordance with eq. (2.50) we first get for $\tilde{B}^{\alpha}_{\beta}$

$$\tilde{B}^{\alpha}_{\beta} = \begin{array}{c} \longrightarrow \alpha \\ \begin{pmatrix} R_z(B_1 \sin\varphi - B_2 \cos\varphi) & -\frac{R_s}{R} B_3 & -R_s(B_1 \sin\varphi - B_2 \cos\varphi) \\ \frac{R}{R_s}(B_3 + R_z(B_1 \cos\varphi + B_2 \sin\varphi)) & 0 & -R\,(B_1 \cos\varphi + B_2 \sin\varphi) \\ \frac{1}{R_s}(R_z^2 + 1)(B_1 \sin\varphi - B_2 \cos\varphi) & \frac{1}{R}(-R_z B_3 + (B_1 \cos\varphi + B_2 \sin\varphi)) & -R_z(B_1 \sin\varphi - B_2 \cos\varphi) \end{pmatrix} \end{array} \begin{array}{c} \\ \downarrow \\ \beta \end{array} .$$

The relationship between the components (B_1, B_2, B_3) in Cartesian coordinates and (B_r, B_φ, B_z) in cylindrical coordinates is given by

$$B_1 = B_r \cos\varphi - B \sin\varphi$$

$$B_2 = B_r \sin\varphi + B \cos\varphi$$

$$B_3 = B_z .$$

Together with eqs. (2.20) and (2.21), which give the relation between B_r, B_z and the magnetic field lines $R(s,z,t)$, we finally obtain for $\tilde{B}^{\alpha}_{\beta}$

$$\tilde{B}^{\alpha}_{\beta} = \begin{array}{c} \longrightarrow \alpha \\ \begin{pmatrix} -R_z B_\varphi & -\frac{S}{R^2} & R_s B_\varphi \\ \frac{S}{R_s^{\,2}}(1 + R_z^{\,2}) & 0 & -\frac{s R_z}{R_s} \\ -\frac{1}{R_s}(1 + R_z^{\,2}) B_\varphi & 0 & R_z B_\varphi \end{pmatrix} \end{array} \begin{array}{c} \\ \downarrow \\ \beta \end{array} . \qquad (2.61)$$

The same procedures yield for $\tilde{B}^{\alpha\beta}$ eq. (2.51)

$$\tilde{B}^{\alpha\beta} = \begin{pmatrix} 0 & -\frac{s}{R^2 R_s^2}(1+R_z^2) & \frac{1}{R_s} B_\varphi \\ \frac{s}{R^2 R_s^2}(1+R_z^2) & 0 & -\frac{s R_z}{R^2 R_s} \\ -\frac{1}{R_s} B_\varphi & \frac{s R_z}{R^2 R_s} & 0 \end{pmatrix}. \tag{2.62}$$

By substituting eq. (2.62) in eq. (2.52) we get for $\tilde{j}^\beta$

$$\tilde{j}^\beta = (\tilde{B}^{\beta\nu})_{,\nu} = \begin{pmatrix} -\frac{\partial}{\partial z}\left[\frac{1}{R_s} B_\varphi\right] - \frac{1}{R_s} B_\varphi GZ \\ -\frac{\partial}{\partial s}\left[\frac{s}{R^2 R_s^2}(1+R_z^2)\right] + \frac{\partial}{\partial z}\left[\frac{sR_z}{R^2 R_s}\right] - \frac{s}{R^2 R_s^2}(1+R_z^2)GS + \frac{sR_z}{R^2 R_s} GZ \\ \frac{\partial}{\partial s}\left[\frac{1}{R_s} B_\varphi\right] + \frac{1}{R_s} B_\varphi GS \end{pmatrix}.$$

Differentiating and rearranging we get

$$\tilde{j}^\beta = (\tilde{B}^{\beta\nu})_{,\nu} = \begin{pmatrix} -\frac{1}{R_s}\left(\frac{\partial}{\partial z} B_\varphi + \frac{R_z}{R} B_\varphi\right) \\ \frac{s}{R^2 R_s^3} L_s[R] \\ \frac{1}{R_s}\frac{\partial}{\partial s} B_\varphi + \frac{B_\varphi}{R} \end{pmatrix} = \begin{pmatrix} \tilde{j}^s \\ \tilde{j}^\varphi \\ \tilde{j}^z \end{pmatrix} \tag{2.63}$$

where

$$L_s[R] = (1 + R_z^2)\left(R_{ss} - \frac{1}{s} R_s\right) - 2 R_s R_z R_{sz} + R_s^2 R_{zz} + \frac{R_s^2}{R} \tag{2.64}$$

is the differential operator applied to $R(s,z,t)$ (eq. (2.24)) for the vacuum magnetic field.

With eqs. (2.61) and (2.62) the Lorentz force $\tilde{F}^{\alpha}_{(B)}(\xi,t)$ is finally obtained according to eq. (2.49), as

$$\tilde{F}^{\alpha}_{(B)} = -\tilde{B}^{\alpha}_{\beta}\,\tilde{j}^{\beta} = -\begin{pmatrix} \frac{1}{R_s}\left\{B_\varphi\left(R_z\frac{\partial}{\partial z} - \frac{1+R_z^{\,2}}{R_s}\frac{\partial}{\partial s}\right)B_\varphi - \frac{B_\varphi^2}{R} + \frac{s^2}{R^2R_s^{\,4}}(1+R_z^{\,2})L_s[R]\right. \\ \frac{s}{R^2R_s}\left(\frac{\partial}{\partial z}B_\varphi + \frac{R_z}{R}B_\varphi\right) \\ B_\varphi\left(\frac{R_z}{R_s}\frac{\partial}{\partial s} - \frac{\partial}{\partial z}\right)B_\varphi - \frac{s^2R_z}{R^2R_s^{\,4}}L_s[R] \end{pmatrix}. \tag{2.65}$$

In the equation of motion (eq. (2.59)) (three scalar equations) there are four velocities $\tilde{v}^s$, $\tilde{v}^\varphi$, $\tilde{v}^z$, and $\tilde{w}^s$. A fourth relation is provided by the equation of motion of the field lines (eq. (2.11)):

$$\frac{\partial}{\partial t}R = v_r + v_z\frac{\frac{\partial F}{\partial z}}{\frac{\partial F}{\partial r}} - \eta\left[r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial F}{\partial r}\right) + \frac{\partial^2 F}{\partial z^2}\right]\Big/\left(\frac{\partial F}{\partial r}\right)$$

where the differential expressions of $F(r,z)$ are rewritten as expressions of $R(s,z)$. With eqs. (2.2), (2.3), and (2.20)

$$\frac{\partial F}{\partial r} = r\,B_z = \frac{s}{R_s}$$

$$\frac{\partial F}{\partial z} = -\,r\,B_r = -\frac{s}{R_s}R_z$$

and with eqs. (2.22) and (2.25)

$$r\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial F}{\partial r}\right) + \frac{\partial^2 F}{\partial z^2} = L_c[F] = -\frac{s}{R_s^{\,3}}L_s[R]$$

we get

$$\frac{\partial}{\partial t}R = v_r - v_z R_z + \frac{\eta}{R_s^{\,2}}L_s[R]\,.$$

The velocity components v_r and v_z still appear in cylindrical coordinates. Transformation to Cartesian coordinates yields

$$v_r(r,\varphi,z) = v^1(x^1,x^2,x^3)\cos\varphi + v^2(x^1,x^2,x^3)\sin\varphi$$

$$v_z(r,\varphi,z) = v^3(x^1,x^2,x^3) .$$

The relation between the velocity components in Cartesian coordinates (KS1) $v^i(x^1,x^2,x^3)$ and magnetic field line coordinates (KS2) $\tilde{v}^\alpha(s,\varphi,z)$ is provided by eq. (2.34):

$$v^i(x,t) = \frac{\partial x^i}{\partial \xi_\alpha}(\tilde{v}^\alpha + \tilde{w}^\alpha) .$$

These transformation relations are used to replace $v_r(r,\varphi,z)$ and $v_z(r,\varphi,z)$ in the above equation:

$$\frac{\partial}{\partial t}R = R_s\,\tilde{v}^s + R_z\,\tilde{v}^z + \frac{\partial R}{\partial t} - \tilde{v}^z R_z + \frac{\eta}{R_s^2}L_s[R]$$

or

$$\tilde{v}^s = -\eta\frac{L_s[R]}{R_s^3} . \qquad (2.66)$$

This reduces the equation of motion of the field lines (eq. (2.11)) to a simple expression for calculating $\tilde{v}^s$ (the velocity of the plasma perpendicular to the field lines). $L_s[R]$ is proportional to the current in the φ-direction. For infinite conductivity $\tilde{v}^s$ vanishes ($\tilde{v}^s \longrightarrow 0$ for $\eta \longrightarrow 0$). The first of the three equations of eq. (2.59) (equation for the s component), which contains both $\frac{\partial \tilde{w}^s}{\partial t}$ and $\frac{\partial \tilde{v}^s}{\partial t}$, should, therefore, be interpreted as the equation of motion of the field lines, the third and fourth terms $(\frac{\partial}{\partial t}\tilde{v}^s + \tilde{v}^\mu(\tilde{v}^s)_{,\mu})$ representing an additional force that

may be regarded as "inertia of diffusion".

2.33 Pressure equation

For the pressure equation (2.53) we get with $\gamma = \frac{5}{3}$

$$\frac{\partial}{\partial t}\bar{p} + \frac{\partial}{\partial s}(\bar{p}\,\tilde{v}^s) + \frac{\partial}{\partial z}(\bar{p}\,\tilde{v}^z) + (\gamma - 1)\,\bar{p}\,(\frac{\partial}{\partial s}\tilde{v}^s + \frac{\partial}{\partial z}\tilde{v}^z) +$$

$$+ \gamma\bar{p}\,(\tilde{v}^s\,GS + \tilde{v}^z\,GZ + DIV\,W) + \qquad (2.67)$$

$$+ (\gamma - 1)\,(\frac{\partial}{\partial s}\tilde{S}^s + \frac{\partial}{\partial z}\tilde{S}^z + \tilde{S}^s\,GS + \tilde{S}^z\,GZ) = 0$$

or

$$\frac{\partial}{\partial t}DP + (\tilde{v}^s\frac{\partial}{\partial s} + \tilde{v}^z\frac{\partial}{\partial z})\,DP + \gamma\,DP(\frac{\partial}{\partial s}\tilde{v}^s + \frac{\partial}{\partial z}\tilde{v}^z) + \qquad (2.68)$$

$$+ (\gamma - 1)\,(RR_s)^{\gamma-1}\,(\frac{\partial}{\partial s}QS^s + \frac{\partial}{\partial z}QS^z) = 0$$

with

$$DP := \bar{p}\,(RR_s)^{\gamma}$$

$$QS^s := \tilde{S}^s\,(RR_s)$$

$$QS^z := \tilde{S}^z\,(RR_s)$$

Note on the thermal current $\tilde{S}^\mu$, $\tilde{S}_\nu$

The contravariant components of the thermal current $\mathcal{S}$ that appear in eq. (2.67) and the components $\tilde{S}_\nu = -\varkappa\frac{\partial}{\partial\xi_\nu}\tilde{T}$, which are covariant by definition as gradient of the temperature $\tilde{T}(\xi,t) = T(x,t) = \frac{p}{\rho}$, are related by the equations

$$\tilde{S}^\mu = g^{\mu\nu}\tilde{S}_\nu$$

$$\tilde{S}_\nu = g_{\nu\mu}\tilde{S}^\mu . \qquad (2.69)$$

In view of the fact that $\tilde{S}^s$ is neglected in Sect. 3.1 we should discuss at this point the different representations of the vector $\mathcal{S}$ (in KS2) with respect to the direction of the base vectors.
The two triples of base vectors are written in a slightly different form for this purpose. We have

$$n_k = \frac{\partial x^i}{\partial \xi_k}$$ the tangents to the coordinate lines

and $$n^i = \frac{\partial \xi^k}{\partial x_i}$$ the gradients on the coordinate surfaces.

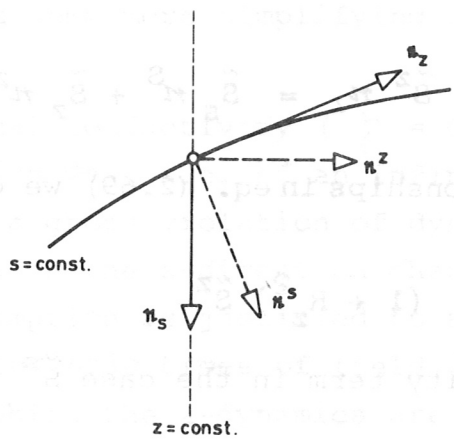


Fig. 3 Base vectors for covariant n_k and contravariant n^k vector components

The thermal current vector $\mathcal{S} = \{S_x, S_y, S_z\}$ can be represented in KS2 in two ways, viz.

$$\mathcal{S} = \tilde{S}_i n^i = \tilde{S}^k n_k .$$

The components in the direction of the base vectors are obtained by forming

$$n^i \tilde{\mathcal{S}} = \tilde{S}^k n_k n^i = \tilde{S}^k \delta_i{}^k = \tilde{S}^i$$

i.e. the $\tilde{S}^i$ are the components of $\tilde{\mathcal{S}}$ in the direction of the gradients of the coordinate surfaces (contravariant vector components).

As the thermal conductivity $\varkappa$ perpendicular to the magnetic field lines is smaller than that parallel to the field lines and, moreover, the temperature gradients in the s-direction are relatively small, the thermal current perpendicular to the coordinate surfaces s = const (i.e. in the direction n^s) is negligible relative to the thermal current in the z-direction ($\tilde{S}^s \ll \tilde{S}^z$).

It will therefore be assumed in the following that

$$\tilde{S}^s \equiv 0$$

i.e.

$$\tilde{\mathcal{S}} = \tilde{S}^z n_z = \tilde{S}_s n^s + \tilde{S}_z n^z .$$

Because of the relationships in eq. (2.69) we get for the case $\tilde{S}^s = 0$

$$\tilde{S}_z = (1 + R_z^2)\, \tilde{S}^z .$$

The thermal conductivity term in the case $\tilde{S}^s = 0$ is thus

$$\frac{\partial}{\partial x_\mu} S^\mu(x,t) = (\tilde{S}^\mu(\xi,t)_{,\mu} = \frac{\partial}{\partial z} \tilde{S}^z + \tilde{S}^z GZ =$$

$$= \frac{\partial}{\partial z} \left(\frac{\tilde{S}_z}{1 + R_z^2}\right) + \frac{\tilde{S}_z}{1 + R_z^2} GZ =$$

$$= -\frac{\partial}{\partial z} \left(\frac{\varkappa}{1 + R_z^2} \frac{\partial}{\partial z} \tilde{T}\right) - \left(\frac{\varkappa}{1 + R_z^2} \frac{\partial}{\partial z} \tilde{T}\right) GZ . \tag{2.70}$$

3. The system of differential equations

In this section first the most essential restrictions on the system of differential equations are discussed and then it is put into the form in which it will finally be solved.

Mathematically speaking, the boundary initial value for the escape of a plasma from a theta pinch shall be consistently formulated.

3.1 The system of equations - further restrictions

To solve the problem some more simplifying assumptions are made at this point.

1) Infinite electrical conductivity ($\eta = 0$):
The assumption of $\sigma = \infty$, i.e. of an infinitely thin current carrying sheath, is a gross violation of dynamics since the motion of the plasma in the z-direction changes in the event of radial diffusion. This assumption is justified to a certain extent, however, if the characteristic times of field diffusion are large relative to those in which the z-dynamics are of interest. An estimate of the diffusion time

$$\Delta t = \frac{L^2}{\eta}$$

where L is the thickness of the current carrying sheath, yields for experiments for which calculations were made

$$\Delta t \approx 10 \ \mu\text{sec} .$$

The times of interest for the z-dynamics, i.e. in which the mass has decreased to $1/e$, are about 2.0 µsec in such cases, this being smaller than the diffusion time.
According to eq. (2.66) $\tilde{v}^s = -\eta L_s [R]/R_s^3$ infinite electrical conductivity $\eta = 0$ means that there is no transport of plasma at all transverse to the field lines, i.e.

$$\tilde{v}^s = 0 \qquad (3.1)$$

2) Thermal current perpendicular to the field lines neglected:

$$\tilde{S}^s = 0 \qquad (3.2)$$

It is obvious that the thermal current perpendicular to the field lines can be neglected when estimating the thermal currents $\tilde{S}^s$ and $\tilde{S}^z$. What is particularly important here is the ratio of the coefficients of thermal conductivity $\varkappa$. The thermal conductivity $\varkappa_{\parallel}$ parallel to the magnetic field is given in **Sect. 1.** The thermal conductivity $\varkappa_{\perp}$ perpendicular to a strong magnetic field is reduced relative to $\varkappa_{\parallel}$ by a factor $1/B^2$ or, to be more precise, by [9]

$$\varkappa_{\perp} = \varkappa_{\parallel} \frac{1}{1 + \frac{6.25}{\sigma^2} \frac{\Omega^2}{\gamma^2}} \qquad \Omega = \frac{e}{m} \frac{B_z}{c} .$$

For the parameters used in the calculations ($B \approx 50$ kG, $kT = 400$ eV, $n = 3 \times 10^{16} cm^{-3}$) we get

$$\frac{\varkappa_{\perp}}{\varkappa_{\parallel}} \approx 10^{-5} .$$

3) No motion of plasma in φ-direction:

$$\tilde{v}^{\varphi} = 0 \qquad (3.3)$$

This means that the plasma does not rotate. A detailed representation of rotational processes in the plasma is obtained in [10].

4) No azimuthal component of the magnetic field:

$$B_{\varphi} = 0 . \qquad (3.4)$$

Owing to the assumption of rotational symmetry ($\frac{\partial}{\partial \varphi} = 0$) this means that no j_z currents flow.

5) Radial equilibrium:

$$\vec{\nabla} p - \vec{j} \times \vec{B} = 0 . \qquad (3.5)$$

For later times of the theta pinch discharge such a condition is tolerably satisfied since the oscillations initiated by fast compression have decayed to a certain extent. Calculation of the magnetic field from this equilibrium condition without allowance for the inertia terms (cf. 3.12) means that the radial oscillations are completely damped. These radial oscillations do not greatly affect the z-motion. Furthermore, the result of this damping is a mean value on which these fast oscillations are superposed. Elimination of these radial oscillations saves a great deal of computing effort.

3.11 Continuity equation

The continuity equation (eq. (2.57) or (2.58)) accordingly reduces to

$$\frac{\partial}{\partial t}\tilde{\rho} + \frac{\partial}{\partial z}(\tilde{\rho}\tilde{v}^z) + \tilde{\rho}\left[\tilde{v}^z \cdot GZ + DIVW\right] = 0 \qquad (3.6)$$

with

$$GZ = \frac{\partial}{\partial z}\ln(RR_s) \qquad (3.7)$$

$$DIVW = \frac{\partial}{\partial t}\ln(RR_s). \qquad (3.8)$$

3.12 Equation of motion and radial equilibrium

After the above simplification the equation of motion (eq. (2.59)) is of the form

$$\begin{pmatrix} \left(\frac{\partial}{\partial t} + 2\tilde{v}^z\frac{\partial}{\partial z}\right)\tilde{w}^s + 2\frac{R_{sz}}{R_s}\tilde{v}^z\tilde{w}^s + \tilde{w}^s\left(\frac{\partial}{\partial s}\tilde{w}^s + \tilde{w}^s\frac{R_{ss}}{R_s}\right) \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} \frac{R_{zz}}{R_s}\tilde{v}^z\tilde{v}^z \\ 0 \\ \left(\frac{\partial}{\partial t} + \tilde{v}^z\frac{\partial}{\partial z}\right)\tilde{v}^z \end{pmatrix} +$$

$$+ \frac{1}{\tilde{\rho}}\begin{pmatrix} \frac{1}{R_s}\left(\frac{1+R_z^2}{R_s}\frac{\partial}{\partial s} - R_z\frac{\partial}{\partial z}\right)\bar{p} \\ 0 \\ \left(-\frac{R_z}{R_s}\frac{\partial}{\partial s} + \frac{\partial}{\partial z}\right)\bar{p} \end{pmatrix} + \frac{1}{\tilde{\rho}}\begin{pmatrix} -\frac{1}{R_s}\frac{s^2(1+R_z^2)}{R^2 R_s^4}L_s[R] \\ 0 \\ \frac{s^2 R_z}{R^2 R_s^4}L_s[R] \end{pmatrix} = 0 \qquad (3.9)$$

with

$$L_s[R] = (1 + R_z^2)(R_{ss} - \frac{1}{s}R_s) - 2R_sR_zR_{sz} + R_s^2 R_{zz} + \frac{R_s^2}{R}. \qquad (3.10)$$

Equation (3.9) is abbreviated

$$\tilde{\varrho}\,(\tilde{C}^{\alpha} + \tilde{A}^{\alpha}) + \tilde{F}^{\alpha} = 0 \tag{3.11}$$

$$\tilde{F}^{\alpha} = \tilde{F}^{\alpha}_{(p)} + \tilde{F}^{\alpha}_{(B)} .$$

As the plasma should always be in radial equilibrium, i.e.

$$\tilde{\varrho}\,\tilde{A}^{S} + \tilde{F}^{S} = 0 \tag{3.12}$$

eq. (3.9) yields for the radial component

$$L_s[R] - \frac{R^2 R_s^3}{s^2}\left\{\frac{\partial}{\partial s}\,\bar{p} + \frac{R_s}{1+R_z^2}\,(R_{zz}\,\tilde{\varrho}\,\tilde{v}^z\,\tilde{v}^z - R_z\,\frac{\partial}{\partial z}\,\bar{p})\right\} = 0 \tag{3.13}$$

as the equation for radial equilibrium (equation of motion in the s-direction). As the individual quantities in eq. (3.13) are time dependent, the equilibrium position shifts (quasi-equilibrium).

The vacuum magnetic field adjoining the plasma (in the radial direction) is described by the differential equation

$$L_s[R] = 0 . \tag{3.14}$$

The equilibrium condition for the plasma boundary (plasma-vacuum interface) is

$$p + \frac{B_p^{\,2}}{2} = \frac{B_v^{\,2}}{2} \tag{3.15}$$

where B_p is the internal magnetic field at the plasma boundary (on the plasma side) and B_v the external magnetic field at the plasma-vacuum interface (on the vacuum side).

With eqs. (2.20) and (2.21) B_p (and B_v) can be expressed in R(s,z)

$$B^2 = B_r^{\,2} + B_z^{\,2} = \frac{s^2}{R^2\,R_s^{\,2}}\,(1 + R_z^{\,2})$$

and hence the equilibrium condition in s,z coordinates

$$2\,p\Big/_{\text{boundary}} + \frac{s^2}{R^2\,R_s^{\,2}}\,(1 + R_z^{\,2})\Big/_{\text{plasma}} = \frac{s^2}{R^2\,R_s^{\,2}}\,(1 + R_z^{\,2})\Big/_{\text{vacuum}} .$$

At the plasma boundary let $s = s_B$. The equilibrium condition for s_B is rearranged as follows:

$$\frac{2\,p}{s^2\,(1 + R_z^{\,2})}\,R^2\Big/_{s_B} + \frac{1}{R_s^{\,2}}\Big/_{\text{plasma}} - \frac{1}{R_s^{\,2}}\Big/_{\text{vacuum}} = 0 \,. \tag{3.16}$$

The equation of motion in the z-direction is

$$\tilde{\varrho}\,(\,\tilde{C}^z + \tilde{A}^z) + \tilde{F}^z = 0 \,. \tag{3.17}$$

From eq. (3.9) this gives us

$$\tilde{\varrho}\,(\frac{\partial}{\partial t} + \tilde{v}^z \frac{\partial}{\partial z})\,\tilde{v}^z - \frac{R_z}{R_s}\frac{\partial}{\partial s}\,\bar{p} + \frac{\partial}{\partial z}\,\bar{p} + \frac{s^2\,R_z}{R^2\,R_s^{\,4}}\,L_s\left[R\right] = 0 \,.$$

If the differential expression $L_s\left[R\right]$ is eliminated with eq. (3.13), the resulting equation of motion in the z-direction is

$$\tilde{\varrho}\left[(1 + R_z^{\,2})\,(\frac{\partial}{\partial t} + \tilde{v}^z \frac{\partial}{\partial z})\,\tilde{v}^z + R_z\,R_{zz}\,\tilde{v}^z\,\tilde{v}^z\right] + \frac{\partial}{\partial z}\,\bar{p} = 0 \,. \tag{3.18}$$

3.13 Pressure equation

The pressure equation (eq. (2.67) or (2.68)) reduces to

$$\frac{\partial}{\partial t}\,\bar{p} + \frac{\partial}{\partial z}\,(\,\bar{p}\,\tilde{v}^z\,) + (\gamma - 1)\,\bar{p}\,\frac{\partial}{\partial z}\,\tilde{v}^z + \gamma\,\bar{p}\,(\,\tilde{v}^z\,GZ + DIVW\,) + $$
$$+ (\gamma - 1)\,(\frac{\partial}{\partial z}\,\tilde{S}^z + \tilde{S}^z\,GZ\,) = 0 \tag{3.19}$$

with

$$GZ = \frac{\partial}{\partial z}\,\ln(RR_s)$$
$$DIVW = \frac{\partial}{\partial t}\,\ln(RR_s)$$
$$\tilde{S}^z = -\frac{\varkappa}{1 + R_z^{\,2}}\,\frac{\partial}{\partial z}\,\tilde{T} \,. \tag{3.20}$$

3.2 Initial conditions and boundary conditions

3.21 Initial conditions

The calculations start with an already compressed plasma in radial equilibrium (eq. (3.12)). Accordingly, we give as initial conditions a radial pressure profile $P(r)$ and a vacuum magnetic field B_o = const(r) such that equilibrium exists.

The initial state should be an equilibrium state, i.e.

$$\tilde{F}^{\alpha}(\xi, t=0) = 0 .$$

From the equation of motion (eq. (3.9)) we thus get the equilibrium conditions

$$\left(\frac{1 + R_z^2}{R_s} \frac{\partial}{\partial s} - R_z \frac{\partial}{\partial z} \right) \bar{p} - \frac{s^2 (1 + R_z^2)}{R^2 R_s^4} L_s [R] = 0 \tag{3.21}$$

$$\left(- \frac{R_z}{R_s} \frac{\partial}{\partial s} + \frac{\partial}{\partial z} \right) \bar{p} + \frac{s^2 R_z}{R^2 R_s^4} L_s [R] = 0 . \tag{3.22}$$

Eliminating $\frac{\partial}{\partial s} \bar{p}$ or $\frac{\partial}{\partial z} \bar{p}$ yields

$$\frac{\partial}{\partial z} \bar{p} = 0 \tag{3.23}$$

$$\frac{\partial}{\partial s} \bar{p} - \frac{s^2}{R^2 R_s^3} L_s [R] = 0 . \tag{3.24}$$

Eq. (3.24) expresses the differential equation for the equilibrium $\vec{\nabla} \cdot P - \vec{j} \times \vec{B} = 0$ in magnetic field coordinates. Eq. (3.23) states that $\bar{p} = \bar{p}(s)$ is constant along a field line. For a given pressure profile $\bar{p} = \bar{p}(s)$ eq. (3.24) (the differential equation for the initial equilibrium in the radial direction) can be integrated, i.e. with given boundary conditions for $R(s,z)$ the distribution of the field lines $R = R(s,z)$ can be calculated.

The problem of giving an arbitrary pressure profile $P = P(r,z)$ in equilibrium with the magnetic field cannot be solved generally since

in the case of equilibrium the function $\bar{p} = \bar{p}(s,z)$ has to satisfy the condition eq. (3.23).

One may, however, proceed as follows:

Let a radial pressure profile $P = P(r)$ be given in a surface $z = z_o$ in which for all r the radial component of the magnetic field $B_r(r,z)$ and its derivative, with respect to z, $\frac{\partial B_r}{\partial z}$ vanish. For this $z = z_o$ it is then possible to calculate the function $\bar{p} = \bar{p}(s)$, and thus the pressure distribution $\bar{p} = \bar{p}(s,z)$ is determined by eq. (3.23).

Under these conditions (i.e. $\bar{B} = \bar{B}(r) = (0,0,B_z(r))$ as plasma magnetic field and $\bar{B}_o = \text{const} = (0,0,B_{zo})$ as vacuum magnetic field) the differential equation $-\vec{\nabla}p + \vec{j} \times \vec{B} = 0$ can be integrated

$$P(r) + \frac{B(r)^2}{2} = \frac{B_o^2}{2} . \tag{3.25}$$

With the relation eq. (2.21) between magnetic field and field lines

$$B_z(r) = \frac{s}{RR_s}$$

we get from eq. (3.25)

$$\left(\frac{s}{RR_s}\right)^2 = B_o^2 - 2\,P(r)$$

or

$$s\,ds = \sqrt{B_o^2 - 2\,P(r)}\;dr . \tag{3.26}$$

For known $P(r)$ the solution of this differential equation yields the relationship

$$r = R = R(s)$$

and hence

$$P = P(r) = P(R(s)) = \bar{p}(s) .$$

With the function $\bar{p} = \bar{p}(s)$ determined in this way it is then possible to solve the differential equation (3.24) for given boundary conditions $R(s_c,z) = R_c(z)$, i.e. to calculate the distribution of the field lines (i.e. the coordinate system)

$$R = R(s,z) .$$

The pressure profile $\bar{p} = \bar{p}(s)$, with $\frac{\partial}{\partial z}\bar{p} = 0$, and the magnetic field determined by $R = R(s,z)$ (eqs. (2.20) and (2.21)) are then in equilibrium. In the event of a small disturbance due to $\frac{\partial}{\partial z} B_r \neq 0$ we can still calculate the equilibrium by iteration. It is assumed that the pressure profile $P = P(r)$ is given by an analytical function (parabola)

$$P(r) = P_o - P_1 (R/R_B)^2 . \tag{3.27}$$

With such a pressure profile the integral of the differential equation (3.26) is obtained by quadrature

$$\frac{1}{2} s^2 = \frac{1}{3b} (a + b R^2)^{3/2} + C \tag{3.29}$$

the abbreviations used being

$$a = B_o^2 - 2 P_o \qquad b = \frac{2 P_1}{R_B^2} . \tag{3.28}$$

Because of $R(s=0) = 0$ we get for the integration constant C

$$C = - \frac{a^{3/2}}{3 b} \tag{3.3o}$$

and hence

$$s^2 = \frac{2}{3 b} ((a + b R^2)^{3/2} - a^{3/2}) . \tag{3.31}$$

This equation gives the relation $R = R(s)$ for the interior of the plasma ($0 \leq r \leq R_B$ or $0 \leq s \leq s_B$).

For the plasma boundary $r = R_B(=RB)$, $s = s_B$ $(=SB)$ eq. (3.31) gives

$$SB = \sqrt{\frac{2}{3 b} ((a + bR^2)^{3/2} - a^{3/2})} \tag{3.32}$$

and hence the step size $DS = \Delta s$ in the s-direction (equidistant)

$$DS = SB/IP \tag{3.33}$$

where IP is the number of intervals in the plasma.

Solving eq. (3.31) for R yields

$$R = R(s) = \sqrt{\frac{a}{b}\left[\left(\frac{3\,b}{2\,a^{3/2}}\,s^2 + 1\right)^{2/3} - 1\right]} \qquad (3.34)$$

the location of the magnetic field lines within the plasma

$$(0 \leq R \leq RB , \quad 0 \leq s \leq SB).$$

The location of the magnetic field lines in the vacuum is determined from the definition of the magnetic field surfaces eq.(2.1) (where transformation from F to s(eq. (2.18)) is used), viz.

$$\frac{1}{2}\,s^2 = F = \int_0^r B_z(r',z,t)\; r'\; dr' . \qquad (3.35)$$

Thus, for the plasma boundary r = RB, s = SB it holds in particular that

$$\frac{1}{2}\,SB^2 = \int_0^{RB} B_z(r',z,t)\; r'\; dr'. \qquad (3.36)$$

For the vacuum $RB \leq R \leq RC$, $SB \leq s \leq SC$ ($RC = R_c$ coil radius, $SC = S_c$ value of s at the coil) the integral eq. (3.35) yields

$$\frac{1}{2}\,s^2 = \int_{r=0}^{RB} B_z\; r\; dr + \int_{r=RB}^{R} B_o\; r\; dr .$$

The first integral is known (eq.(3.36)) and the second can readily be evaluated because of B_o = const(r):

$$s^2 = SB^2 + B_o\,(R^2 - RB^2) . \qquad (3.37)$$

This is (analogous with eq.(3.31)) the relationship R = R(s) for the vacuum ($RB \leq r \leq RC$, $SB \leq s \leq SC$).

For R = RC we obtain the value of SC

$$SC = \sqrt{SB^2 + B_o(RC^2 - RB^2)} \qquad (3.38)$$

and hence of the (equidistant) step size in the vacuum $DSV = \Delta s_v$

$$DSV = (SC - SB) / IV$$

where IV is the number of intervals in the vacuum.

Solving eq. (3.37) for R gives

$$R = R(s) = \sqrt{RB^2 + \frac{1}{B_o}(s^2 - SB^2)} \qquad (3.40)$$

the location of the magnetic field lines in the vacuum

$$(RB \leq R \leq RC \ , \quad SB \leq s \leq SC) \ .$$

Given the plasma radius R_B, coil radius R_c, vacuum magnetic field $B_o = (o, o, B_{zo})$, and a pressure profile $P = P(r)$ at $z = z_o$, a plasma magnetic field and the coordinate system $R = R(s)$ are calculated first for $z = z_o$ by means of the given equations. With given boundary conditions $R(s_c, z) = R_c(z)$ the solution of the differential equation (3.24) gives the coordinate system

$$R = R(s,z) \ ,$$

in such a way that equilibrium between the magnetic field (determined by $R = R(s,z)$) and the gas kinetic pressure $p = p(s,z)$, with $\frac{\partial}{\partial z} p(s) = 0$, is ensured.

The two coefficients P_o and P_1 of the pressure parabola can be expressed by the corresponding values of the particle density n_o, n_1 and temperature kT_o, kT_1. A parabola profile is given for the density $n(r) = n_o - n_1 (R/R_B)^2$. Together with a given temperature $kT(r) = kT_o$, it is then possible to determine a pressure profile

$$P(r) = n_o \, kT_o - n_1 \, kT_o \, (R/R_B)^2$$

$$P_o = n_o \, kT_o \ , \quad P_1 = n_1 \, kT_o \ .$$

If P_1 is now changed in proportion to a given temperature profile by

$$P_1 \Longrightarrow P_1 + (n_o - n_1) \cdot kT_1 \ ,$$

radial parabola profiles can be given as initial conditions for both the density and temperature. (The temperature $kT(r)$ is calculated from this pressure profile and the given density profile by $kT(r) = P(r) / n(r)$.)

3.22 Boundary conditions

To solve the problem we should really have to calculate the entire external space as well. This is extremely difficult not only because of the complicated physical conditions involved, but also practically impossible because of the limited storage capacity of a computer.

The processes in this external space, however, exert only a slight influence on the interior of the plasma since the external space is practically a vacuum.

The assumption of a periodic continuation of the magnetic field means that for the field lines $R = R(s,z,t)$ there is a vanishing derivative with respect to z in the end plane

$$\left.\frac{\partial R}{\partial z}\right/_{\text{end plane}} = 0 \quad . \tag{3.41}$$

For the boundary conditions of the dynamic quantities $\tilde{\rho}$, $\tilde{\rho}\cdot\tilde{v}^z$, $\bar{p}$ and the thermal current $\tilde{S}^z$ it was assumed that the plasma that has escaped through the end plane should have no after-effects whatsoever on the rest of the plasma, i.e. the plasma should adjoin an infinitely large vacuum. When difference equations are formulated in integral form where flows through the boundaries of volume elements are written by interpolating the neighbouring quantities (within and beyond these volume elements), this means that for the last volume element this flow is expressed only by the internal quantities. This amounts to an extrapolation. As the flows are calculated by averaging the values of the quantities in the boundaries (linear interpolation), the extrapolation at the ends is linear. That is, the boundary conditions for the dynamic quantities $Y: \left\{\tilde{\rho}, (\tilde{\rho}\cdot\tilde{v}^z), \bar{p}, \tilde{S}^z\right\}$ are:

$$\left.\frac{\partial^2}{\partial z^2} Y\right/_{\text{end plane}} = 0 \quad . \tag{3.42}$$

3.3 Brief formulation of the initial boundary value problem with radial quasi-equilibrium

System of equations

In the following the symbol $\sim$ denoting that the quantity A: $\{\rho, \varkappa, p, T\}$ is meant in KS2 (i.e. $A = \tilde{A}(\xi, t)$) is omitted. We thus have

$$\tilde{\rho} = \rho \qquad \tilde{v}^z = V \qquad \bar{p} = P \qquad \tilde{T} = T .$$

The system of equations is then written

$$\frac{\partial}{\partial t}\rho + \frac{\partial}{\partial z}(\rho \cdot V) + \rho\left[V \cdot GZ + DIVW\right] = 0 . \tag{3.43}$$

$$\left(\frac{\partial}{\partial t} + V\frac{\partial}{\partial z}\right) V + \frac{R_z R_{zz}}{1+R_z^2} V^2 + \frac{1}{1+R_z^2}\frac{1}{\rho}\frac{\partial}{\partial z} P = 0 . \tag{3.44}$$

$$\begin{aligned} &\frac{\partial}{\partial t} P + \frac{\partial}{\partial z}(P \cdot V) + (\gamma-1) P \frac{\partial}{\partial z} V + \gamma P\left[V \cdot GZ + DIVW\right] + \\ &- (\gamma-1)\frac{\partial}{\partial z}\left(\frac{\varkappa}{1+R_z^2}\frac{\partial}{\partial z} T\right) - (\gamma-1)\left(\frac{\varkappa}{1+R_z^2}\frac{\partial}{\partial z} T\right) GZ = 0 . \end{aligned} \tag{3.45}$$

$$L_s[R] - \frac{R^2 R_s^3}{s^2}\left\{\frac{\partial}{\partial s} P + \frac{R_s}{1+R_z^2}\left(R_{zz}\rho V^2 - R_z \frac{\partial}{\partial z} P\right)\right\} = 0 \qquad (0 \leq S \leq SB) \tag{3.46}$$

$$L_s[R] = 0 \qquad (SB < S \leq SC) \tag{3.47}$$

$$\frac{2P}{s^2(1+R_z^2)}\Bigg/ R_s^2 - \left[\frac{1}{R_s^2}\right]\Bigg/_{VAC} + \left[\frac{1}{R_s^2}\right]\Bigg/_{PLA} = 0 \qquad (S = SB) \tag{3.48}$$

where

$$GZ = \frac{\partial}{\partial z}\ln(RR_s) \tag{3.49}$$

$$DIVW = \frac{\partial}{\partial t}\ln(RR_s) \tag{3.50}$$

$$L_s[R] = (1+R_z^2)\left(R_{ss} - \frac{1}{s}R_s\right) - 2R_s R_z R_{sz} + R_s^2 R_{zz} + R_s^2/R . \tag{3.51}$$

Initial conditions

Let z-independent initial distributions of the particle density $n = n(r)$ and temperature $kT = kT(r)$, i.e. a radial pressure profile $P = P(r)$ in the form of a parabola

$$P(r,t=0) = P_o - P_1 \left(\frac{r}{R_B}\right)^2$$

and a vacuum magnetic field B_o constant in r and z be given. With eqs. (3.34) and (3.35) we get the location of the field lines for $t = 0$. This distribution is in equilibrium. Let the velocity be $V = V(r,z,t=0) = 0$.

Boundary conditions

The periodic continuation of the magnetic field B is given by

$$\left.\frac{\partial R}{\partial z}\right|_{\text{end plane}} = 0.$$

The free outflow of the quantities $Y : \{\rho, (\rho \cdot V), P, q\}$ is described by

$$\left.\frac{\partial^2 \xi}{\partial z^2} Y\right|_{\text{end plane}} = 0.$$

4. Results

Calculations were made for the plasma configurations in the ISAR III theta pinch experiment /12/.

This plasma was chosen because the fluid theory can be applied to it with relative ease and because, moreover, measurements of end losses in this very experiment were already in progress.

The essential data for the calculations are:

Coil length	$2 \times ZL$	=	30 cm
Coil radius	R_{coil}	=	3.5 cm
Plasma radius	R_B	=	0.4 cm
Mean field	$\langle B_{max} \rangle$	=	70 kG

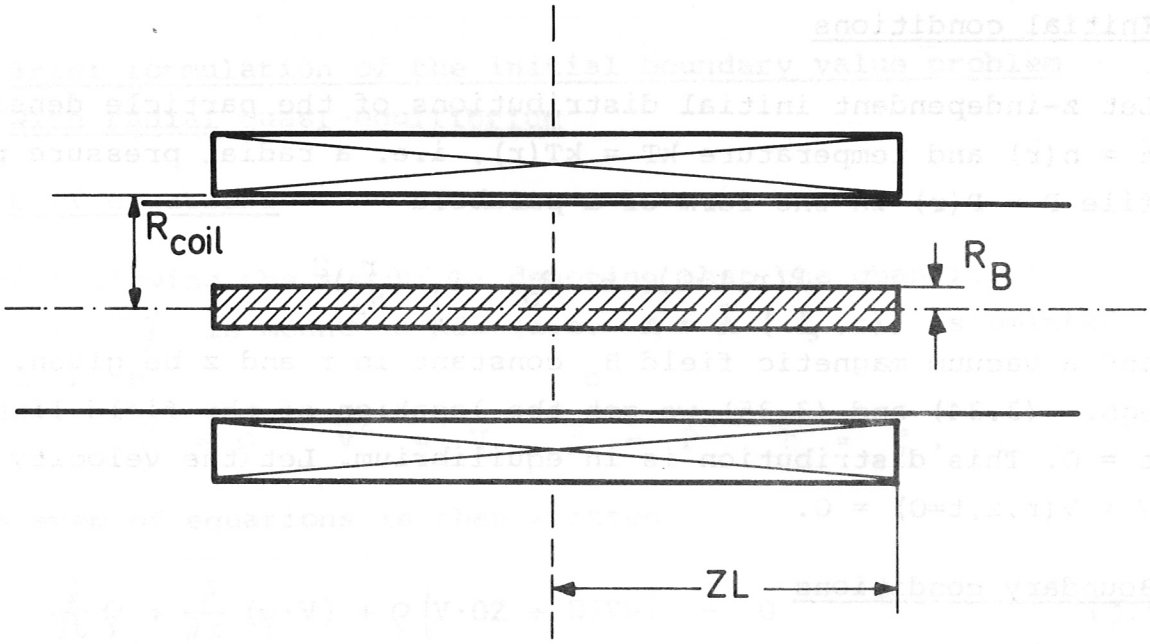


Fig. 4 The most important data for the calculations are:
R_B = 0.4 cm, T_{coil} = 3.5 cm, ZL = 15 cm, $\langle B_{max} \rangle$ = 70 kG.

As it is assumed in the calculation that the $\dot{B}$ in the coil vanishes, a mean maximum B was taken. The measured half-width of 0.4 cm was used as plasma radius R_B (also regarded as a value averaged over time).

The radial profiles for the initial density and initial temperature (for the calculations) are plotted in Fig. 5

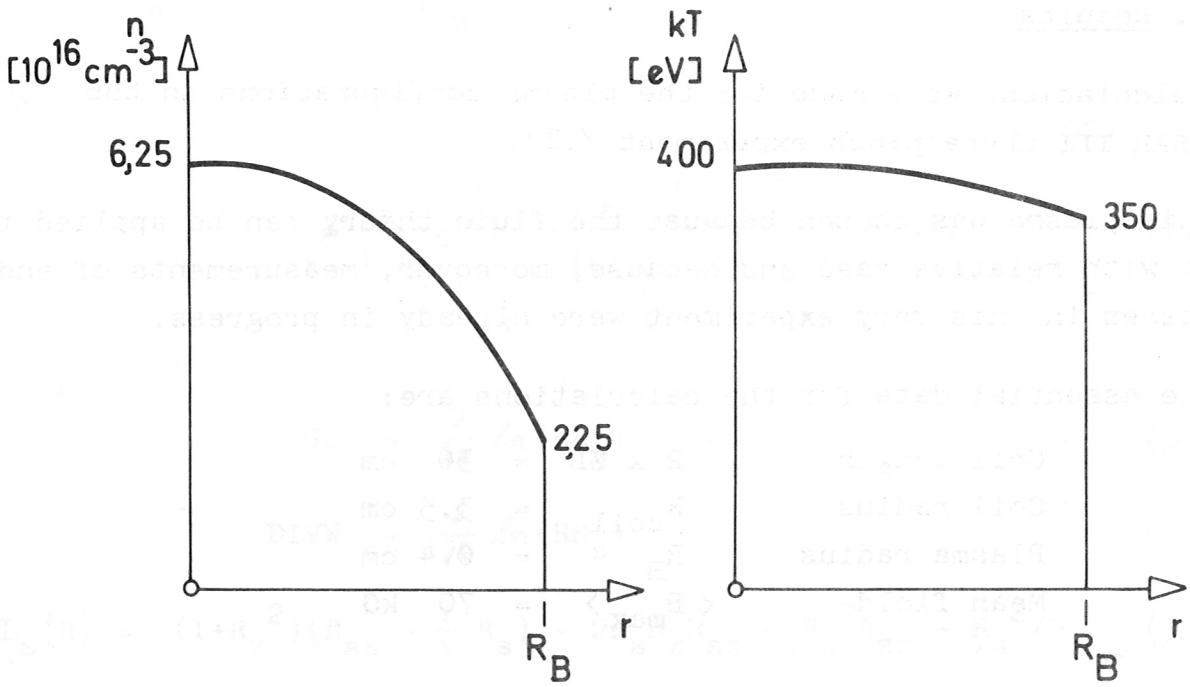


Fig. 5 Radial initial distributions of particle density
n = n(r) and temperature kT = kT(r).

4.1 Influence of thermal conduction on end losses

4.11 Mass losses

The decrease of the total mass m over time t, normalized to the total mass m_o at the start of the calculation (t = 0), is plotted in Fig.6, first for calculations allowing for thermal conduction ($\varkappa > 0$) and then neglecting thermal conduction ($\varkappa = 0$).

The mass decreases approximately exponentially with time. A characteristic for the mass decrease is the containment time τ_M, this being the time taken by the mass to decrease to 1/e of the initial mass (e-folding time).

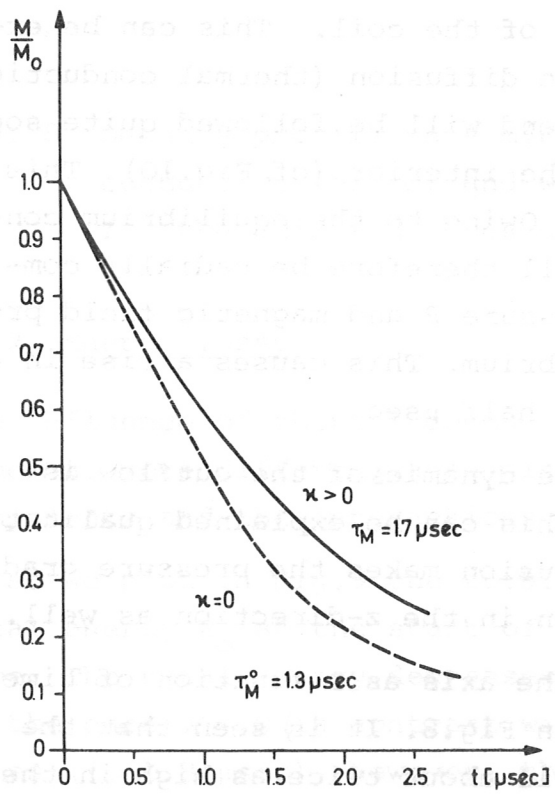


Fig.6

Relative mass decrease M(t) / M(t=0) allowing for ($\varkappa > 0$) and neglecting ($\varkappa = 0$) thermal conduction.

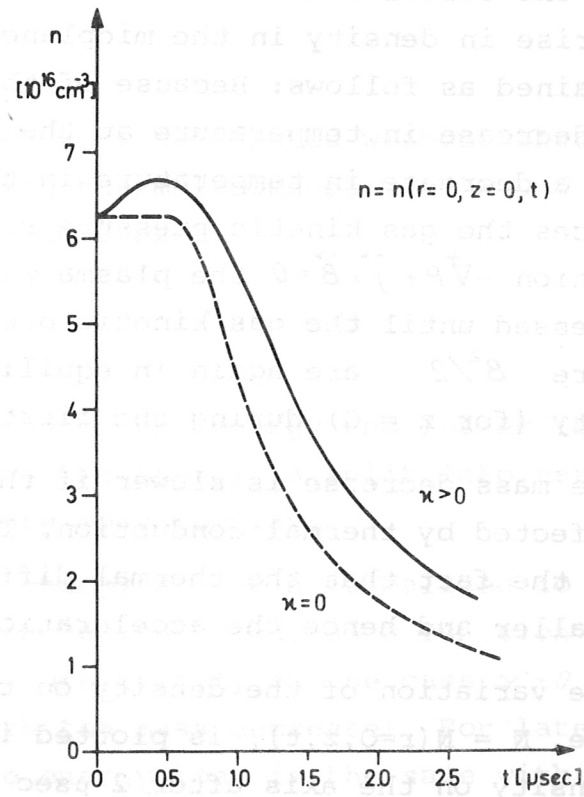


Fig.7

Time variation of the particle density n in the centre of the vessel (r=0, z=0) for $\varkappa > 0$ and $\varkappa = 0$.

The influence of thermal conduction is shown by comparing the τ_M values:

τ_M^o = 1.3 μsec for the case $\varkappa = 0$

τ_M = 1.7 μsec for the case $\varkappa > 0$.

The particle density n in the centre of the vessel (r = 0, z = 0) is plotted in Fig.7 as a function of time. This reveals the influence of thermal conduction on the local mass decrease.

In the case without thermal conduction ($\varkappa = 0$) the region in the centre of the plasma remains undisturbed until reached by the rarefaction wave due to the outflow of plasma at the end. This happens after about 0.7 μsec, the time taken by the wave to propagate from the end to the midplane with the speed of sound ($c^2 = \gamma \frac{P}{\rho}$).

If the calculations allow for thermal conduction, first we observe a rise in density in the midplane of the coil. This can be explained as follows: Because of the diffusion (thermal conduction) a decrease in temperature at the end will be followed quite soon by a decrease in temperature in the interior (cf. Fig.10). This reduces the gas kinetic pressure P. Owing to the equilibrium condition $-\vec{\nabla} P + \vec{j} \times \vec{B} = 0$ the plasma will therefore be radially compressed until the gas kinetic pressure P and magnetic field pressure $B^2/2$ are again in equilibrium. This causes a rise in density (for z = 0) during the first half μsec.

The mass decrease is slower if the dynamics of the outflow is affected by thermal conduction. This can be explained qualitatively by the fact that the thermal diffusion makes the pressure gradients smaller and hence the acceleration in the z-direction as well.

The variation of the density on the axis as a function of time, i.e. N = N(r=0,z,t), is plotted in Fig.8. It is seen that the density on the axis after 2 μsec is about twice as high in the case of finite thermal conductivity as in the case of $\varkappa = 0$.

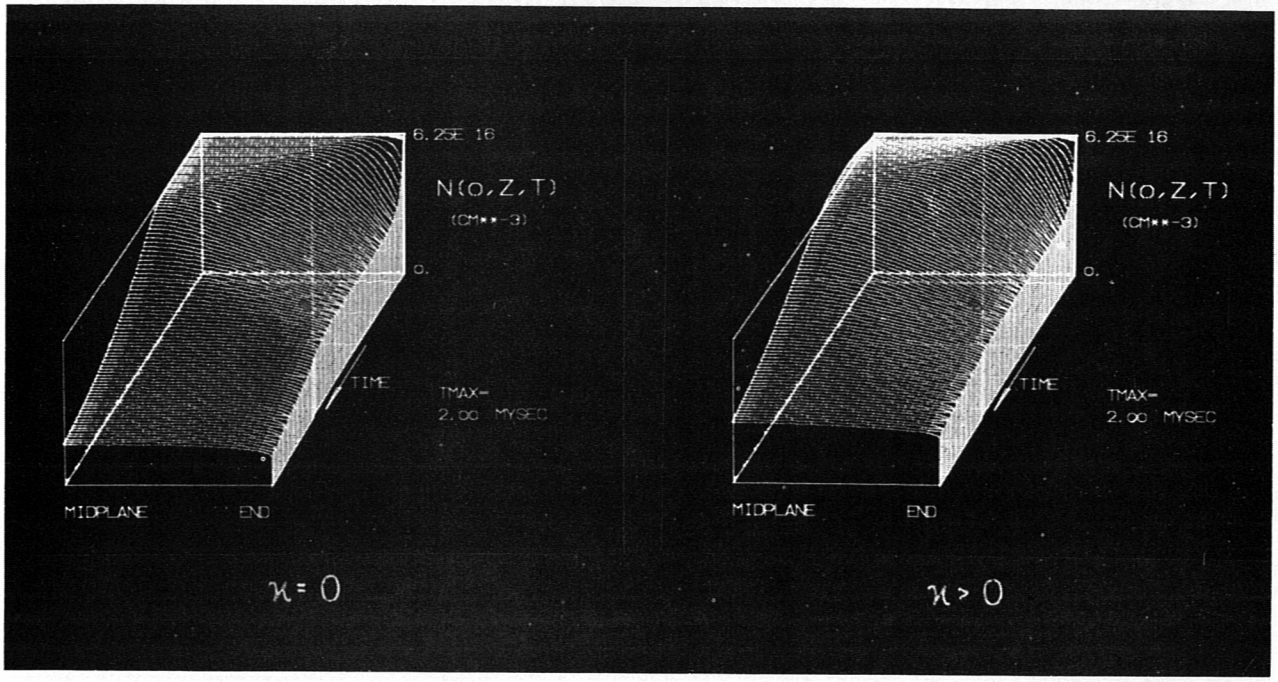


Fig. 8 Density profiles n = m(r=0,z,t) calculated without thermal conduction ($\varkappa = 0$) and with thermal conduction ($\varkappa > 0$) for the axis and times up to 2 μsec.

4.12 Energy losses

The influence of thermal conduction on the energy end losses is somewhat more differentiated since the energy is split into various components and the interactions are more complex.

First we plot in Fig.9 the total energy E, again normalized to the total energy E_o at the start of the calculation, as a function of time. The total energy decreases more quickly in the case $\varkappa > 0$ than in the case $\varkappa = 0$ (in contrast with the mass decrease). For later times ($t > 1.8$ μsec), however, the energy loss is the same within the calculating error of approx. 1 %.

The local thermal energy for r = 0, z = 0 is plotted in Fig. 10 as a function of time. For times $t > 1$ μsec kT is about twice as high in the case $\varkappa = 0$ as in the case with thermal conduction.

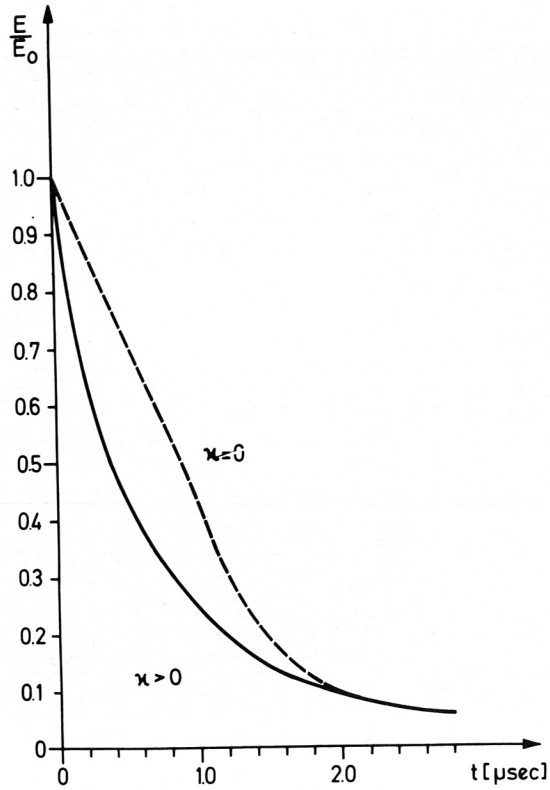


Fig.9
Relative energy decrease E(t) /E(t=0) with ($\varkappa > 0$) and without ($\varkappa = 0$) allowance for thermal conduction

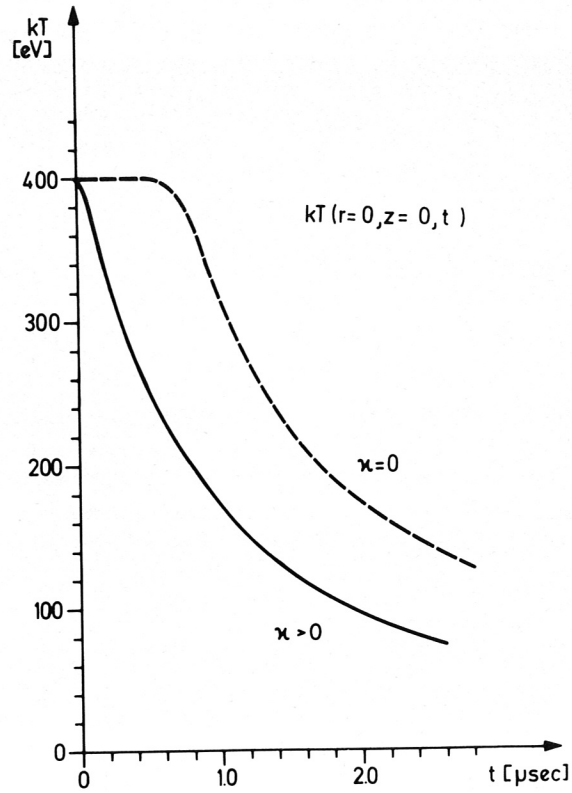


Fig.10
Time development of the temperature kT in the centre of the vessel (r = 0, z = 0) for $\varkappa > 0$ and $\varkappa = 0$.

In Figs. 11 and 12 the various components of the energy losses are plotted for the two cases with and without thermal conduction.

The total energy of the plasma is composed of

$$E = E_{kin} + E_{therm}$$

where $E_{kin} = \iiint \frac{1}{2} \varrho V^2 \, d^3x$ kinetic energy,

$E_{therm} = \iiint \frac{3}{2} P \, d^3x$ thermal energy.

The losses of kinetic energy are obtained by calculating the energy transport through the end plane

$$\Delta E_{kin} = \iiint (\tfrac{1}{2} \varrho V^2) V dt \, d^2f \, .$$

The losses of thermal energy, first by convection and then, in the case $\varkappa > 0$ (with thermal conduction), by diffusion as well, are calculated as follows:

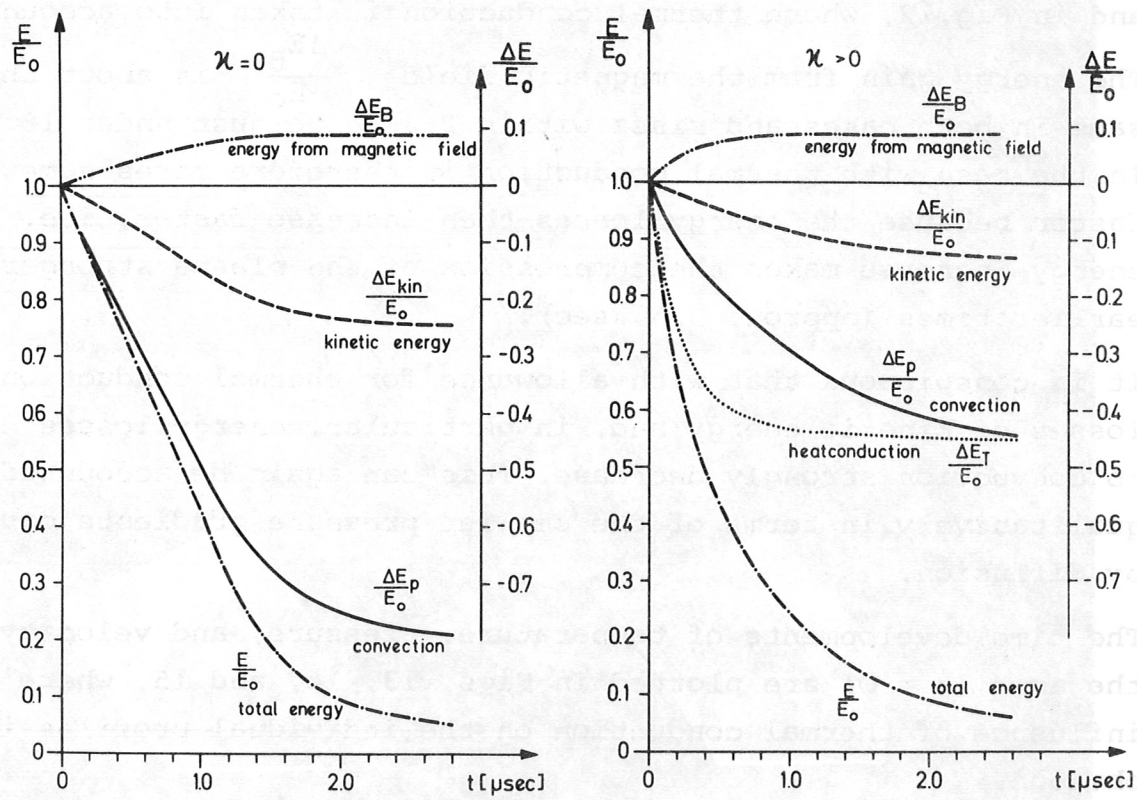
$$\Delta E_p = \iiint \left(\frac{5}{2} P \right) V \, dt \, d^2f$$

$$\Delta E_T = \iiint (-q) \, dt \, d^2f$$

where $q = \varkappa \frac{\partial}{\partial z} T$ is the thermal current.

These energy components of the plasma are now in interaction with the energy content of the magnetic field

$$E_{magn} = \iiint \frac{B^2}{2} d^3x \, .$$



Figs.11 and 12 Relative energy losses calculated with ($\varkappa > 0$) and without ($\varkappa = 0$) thermal conduction.

The total energy of the magnetic field is composed of the energy of the vacuum field E_{B_v} and the energy of the magnetic field in the plasma E_{B_p}, i.e.

$$E_{magn} = E_{B_v} + E_{B_p} .$$

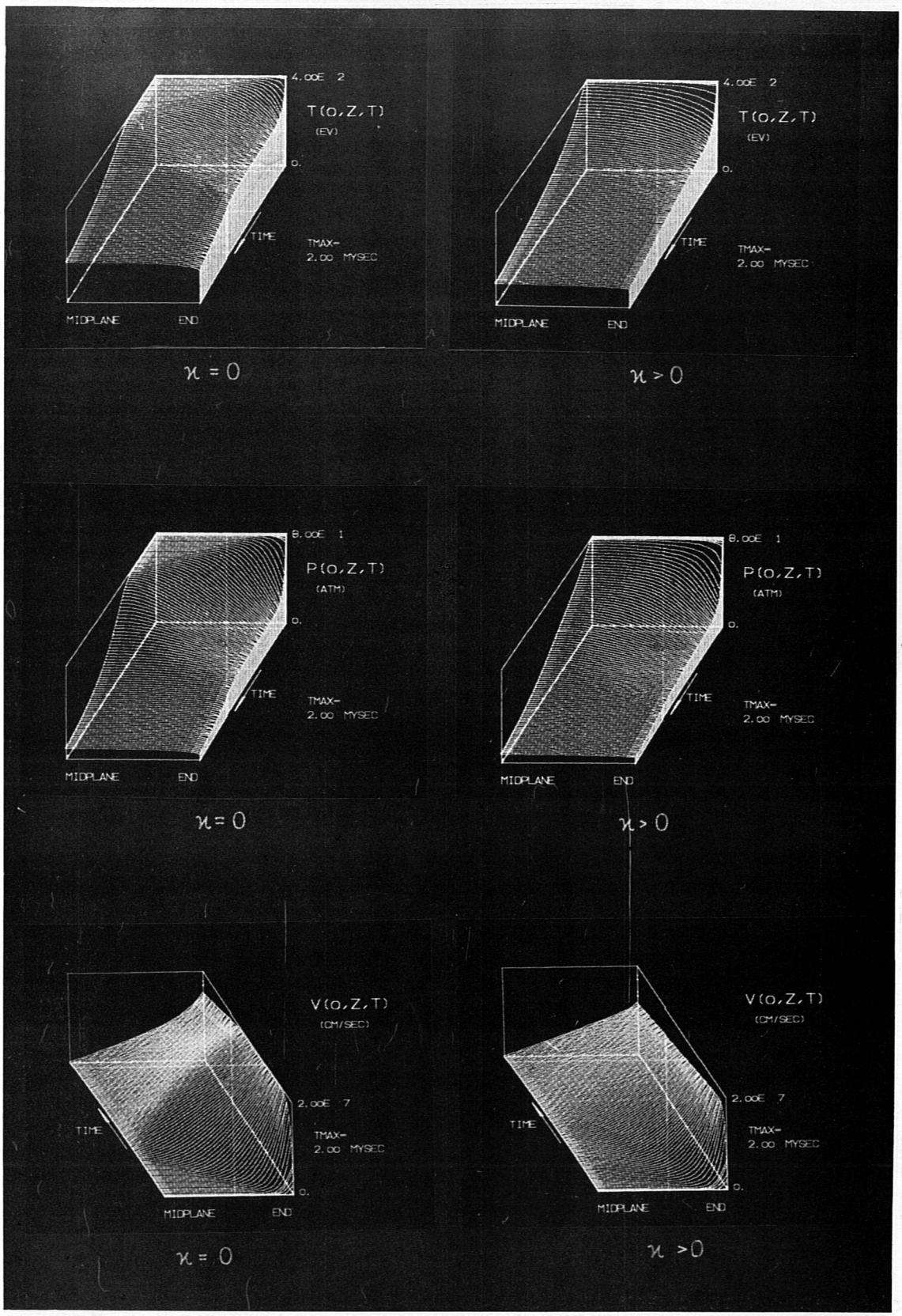
This quantity is constant. The energy gain of the plasma from the magnetic field is thus

$$\Delta E_B \Big/_{t=t_1} = E_{magn} \Big/_{t=0} - E_{magn} \Big/_{t=t_1} .$$

The energy variations $\Delta E_B, \Delta E_{kin}, \Delta E_p, \Delta E_T$ are all normalized to the total energy of the plasma at the beginning of the calculation (t = 0). The time development of the individual components is plotted in Fig.11, where the thermal conduction is neglected ($\varkappa = 0$), and in Fig.12, where thermal conduction is taken into account. The energy gain from the magnetic field $\frac{\Delta E_B}{E_0}$ is about the same in both cases and rises within 2 µsec to just under 10 %. In the case with thermal conduction it therefore rises somewhat faster because the energy losses then increase faster, i.e. the energy increase makes the compression of the plasma stronger at earlier times (approx. 0.5 µsec).

It is conspicuous that with allowance for thermal conduction the losses of kinetic energy and, in particular, energy losses due to convection strongly decrease. This can again be accounted for qualitatively in terms of the smaller pressure gradients caused by diffusion.

The time developments of temperature, pressure, and velocity on the axis (r = 0) are plotted in Figs. 13, 14, and 15, where the influence of thermal conduction on the individual profiles is obvious.



Figs. 13,14,15 z-t profiles for temperature kT(r=0,z,t), pressure P(r=0,z,t), and velocity V(r=0,z,t) for the calculations with ($\varkappa > 0$) and without ($\varkappa = 0$) thermal conduction.

4.2 Dependence of mass losses on the initial parameters

Studying the influence of the initial values on the end losses would require an enormous amount of computing time for reasonable ranges of parameters. The calculations were therefore made with a one-dimensional model that treats only the z-dynamics and neglects the radial dependence of the profiles and the influence of the magnetic field. The result of varying the initial density n_o and initial temperature kT_o for the characteristic containment time τ_M for the mass is plotted in Fig. 18.

The influence of the coil length ZL for τ_M is shown in Fig. 19. These values were also calculated with the one-dimensional programme.

The radial dependence was shown by calculating the initial distributions as in Fig. 16.

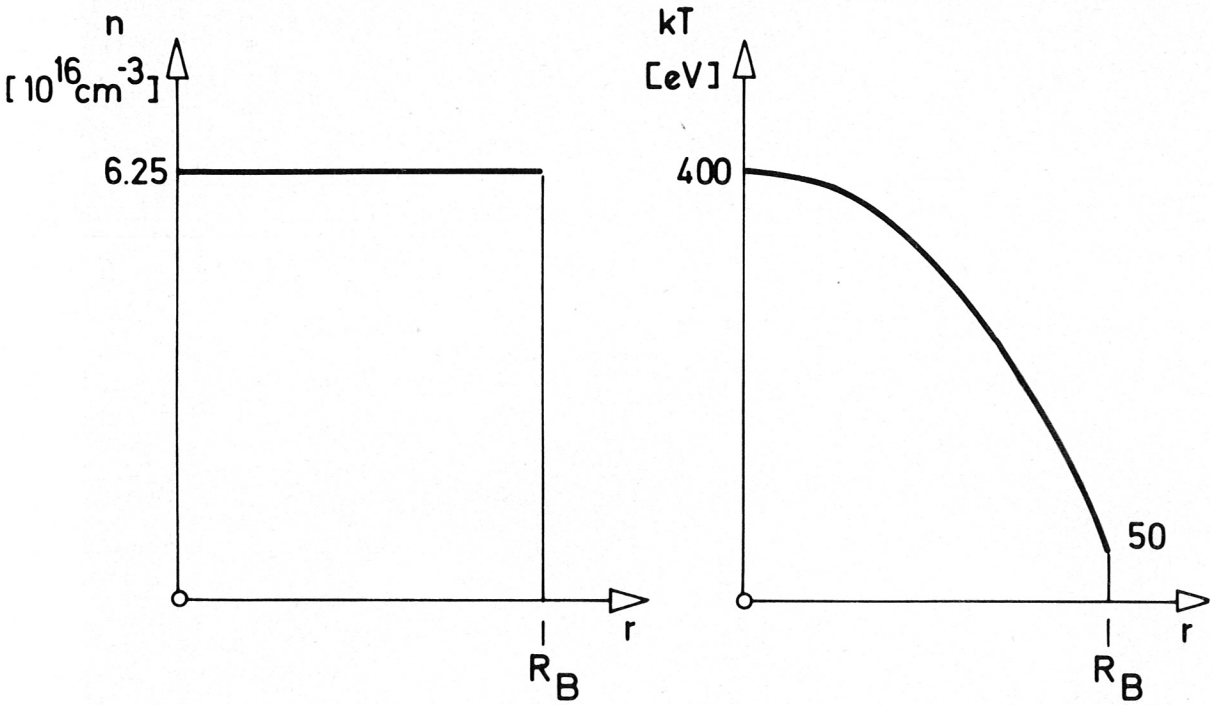


Fig. 16 Initial radial distributions

An r-independent distribution of the particle density n_o and a relatively strong temperature drop in the radial direction means that the speed of sound is much higher on the axis and hence the outflow is much stronger than near the plasma boundary. As a result

the initially r-independent density profile has a distinctly positive gradient in the r-direction after a time (t > 1 μsec) (Fig.17).

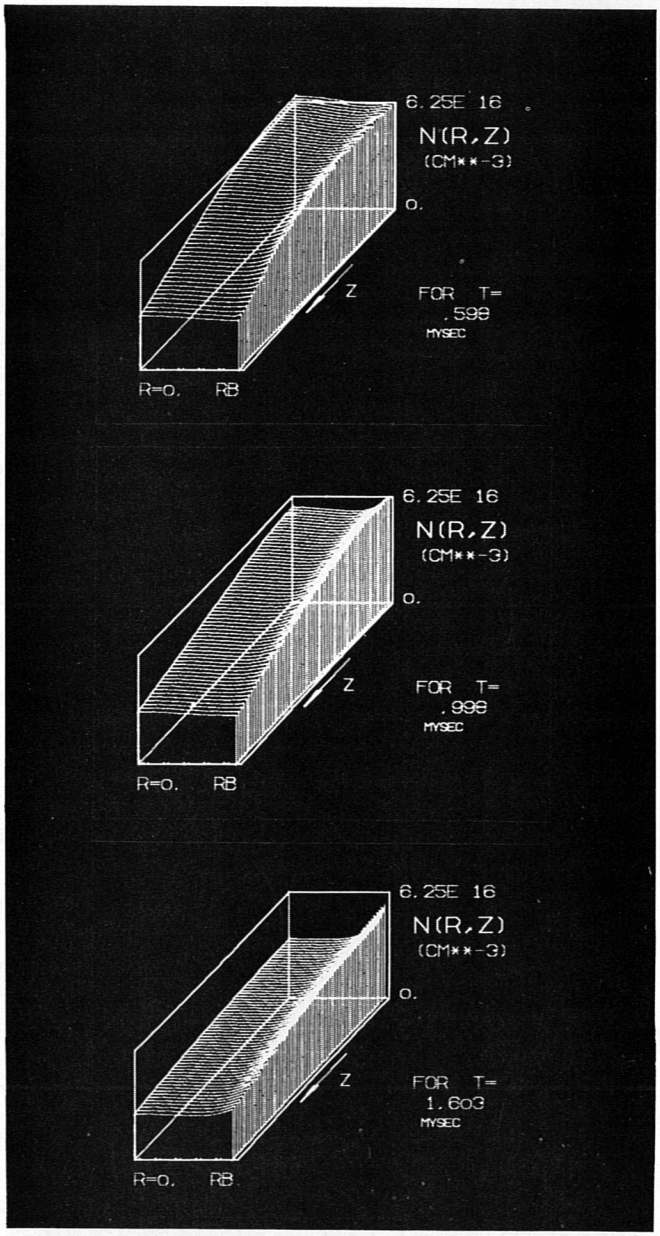


Fig. 17 Density profiles n(r,z) for the initial distributions in Fig. 16 after 0.6, 1.0, and 1.6 μsec.

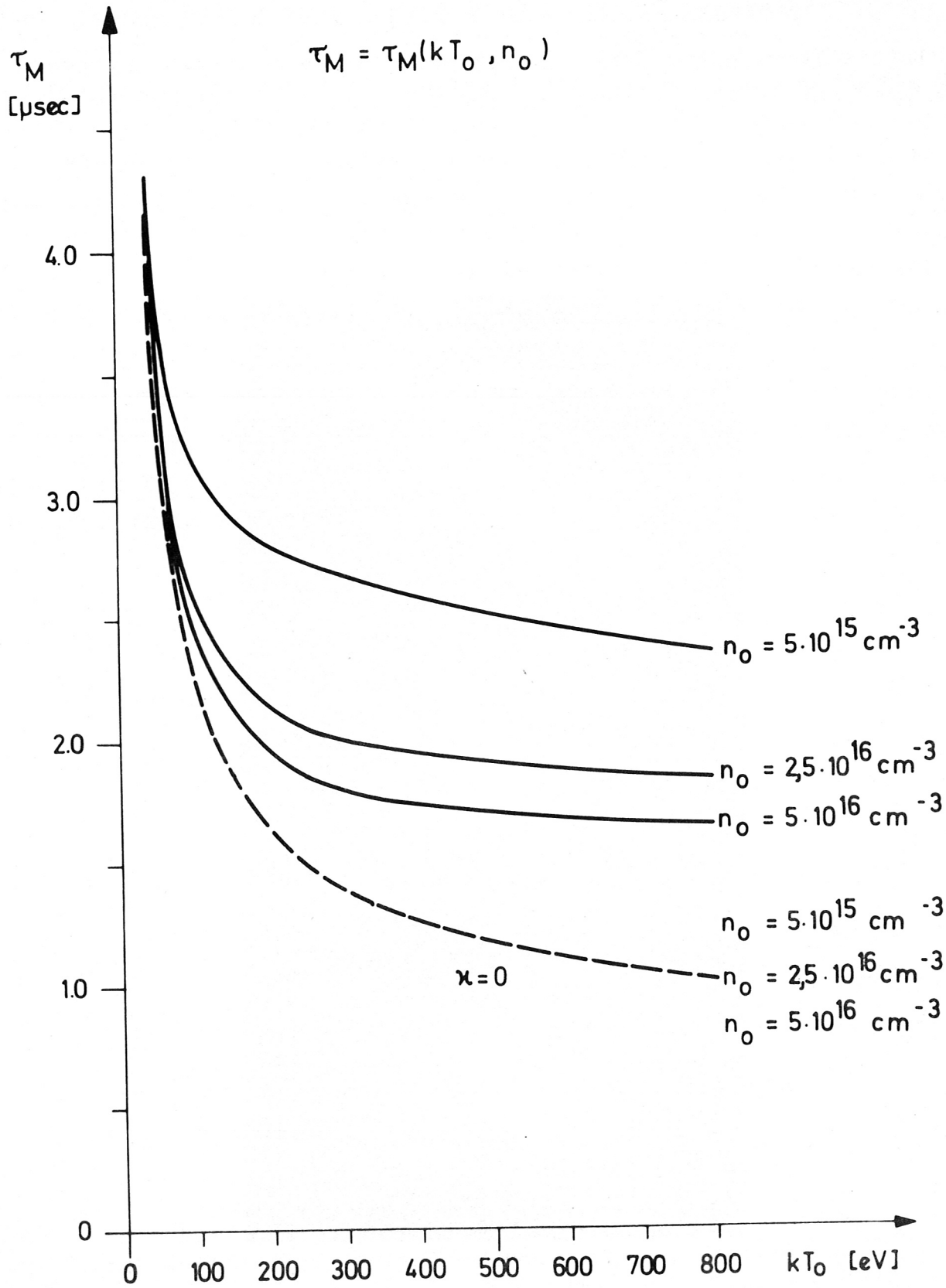


Fig. 18 Dependence of the characteristic containment time τ_M on the initial densities n_o and initial temperatures kT_o .

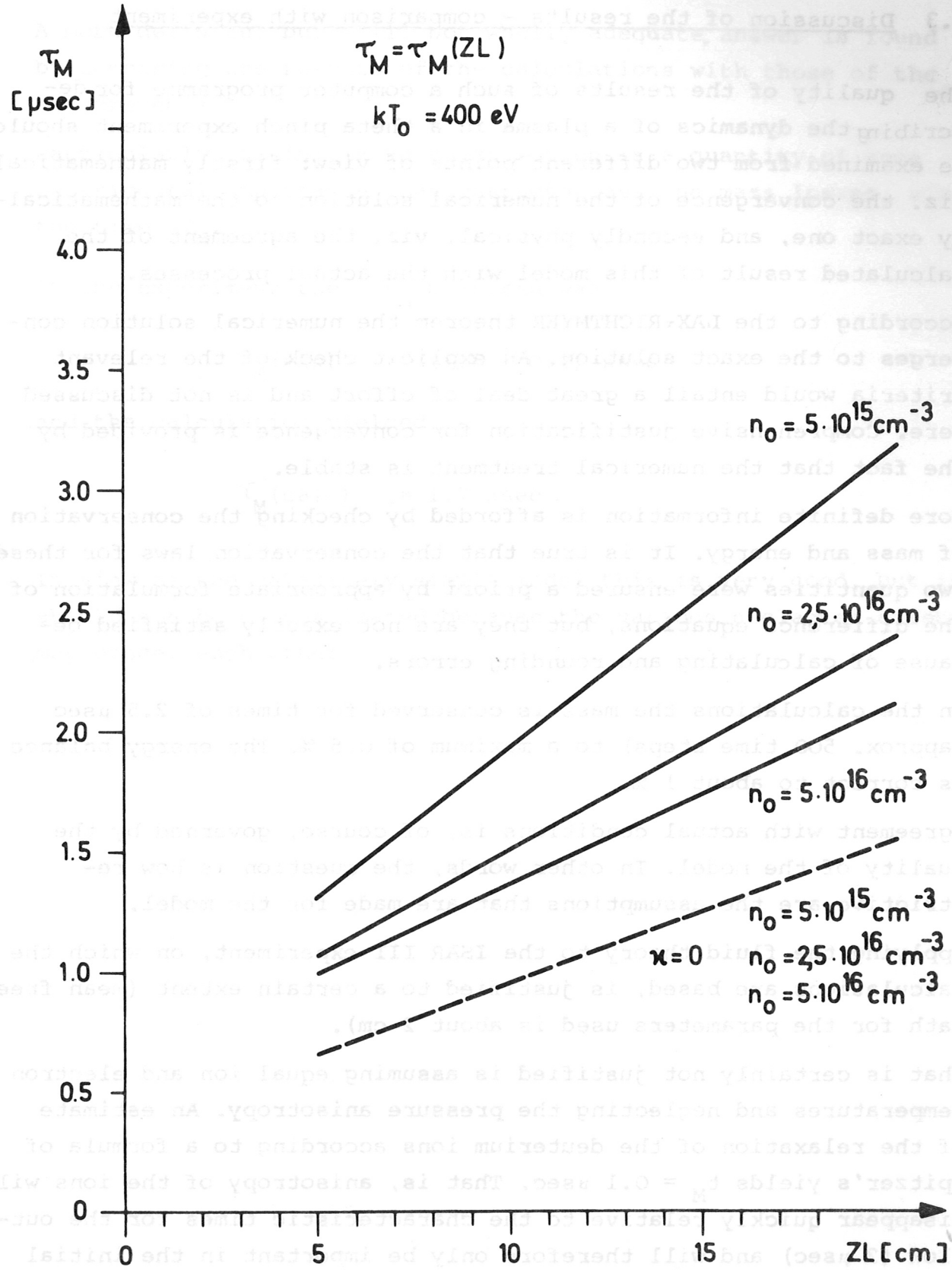


Fig. 19 Dependence of the containment time τ_M on the coil length (ZL is half the coil length) and the initial density n_0 .

4.3 Discussion of the results - comparison with experiment

The quality of the results of such a computer programme for describing the dynamics of a plasma in a theta pinch experiment should be examined from two different points of view: firstly mathematical, viz. the convergence of the numerical solution to the mathematically exact one, and secondly physical, viz. the agreement of the calculated result of this model with the actual processes.

According to the LAX-RICHTMYER theorem the numerical solution converges to the exact solution. An explicit check of the relevant criteria would entail a great deal of effort and is not discussed here. Comprehensive justification for convergence is provided by the fact that the numerical treatment is stable.

More definite information is afforded by checking the conservation of mass and energy. It is true that the conservation laws for these two quantities were ensured a priori by appropriate formulation of the difference equations, but they are not exactly satisfied because of calculating and rounding errors.

In the calculations the mass is conserved for times of 2.5 μsec (approx. 500 time steps) to a maximum of 0.5 %. The energy balance is correct to about 1 %.

Agreement with actual conditions is, of course, governed by the quality of the model. In other words, the question is how restrictive are the assumptions that are made for the model.

Applying the fluid theory to the ISAR III experiment, on which the calculations are based, is justified to a certain extent (mean free path for the parameters used is about 2 cm).

What is certainly not justified is assuming equal ion and electron temperatures and neglecting the pressure anisotropy. An estimate of the relaxation of the deuterium ions according to a formula of Spitzer's yields t_M = 0.1 μsec. That is, anisotropy of the ions will disappear quickly relative to the characteristic times for the outflow (2 μsec) and will therefore only be important in the initial phases of the calculation.

A more definite, but still not wholly adequate, answer is found by comparing the results of the calculations with those of the measurements.

Particularly useful in this connection is a quantity of some experimental importance that characterizes the mass losses, viz. the time τ_M.

In the experiment the τ_M measured was

$$\tau_M(\text{exp}) = (1.8 \pm 0.2)\ \mu\text{sec}$$

and the calculation yielded

$$\tau_M(\text{calc}) = 1.7\ \mu\text{sec}\ .$$

In view of the relatively simple model this is very good, but it should not be overestimated because the various neglected effects may cancel each other.

Summary - further problems

The purpose of this study has been to write a computer programme describing the dynamics of a theta pinch as a function of two space variables (r and z) and the time for a relatively simple plasma model. This has been achieved. In particular, the end losses of mass and energy were determined. The agreement with experimental results (especially for mass losses) is very good as far as this first approximation of the programme is concerned.

Scope for extending the programme and its applications is afforded mainly by three problems:

1) Generalization of the physical model:
   That is, other effects will be taken into account. First the calculations will be extended to a two-fluid model (i.e. $T^{(i)} \neq T^{(e)}$) and the anisotropy of the ion pressure will be allowed for. This point is of particular interest because marked differences between the ion and electron temperatures have often been observed in experiments.

2) Generalization of the initial state:
   It should be possible to give any pressure profile $P = P(r)$ (calculated from arbitrary density and temperature profiles):

$$P(r) = n(r) \cdot kT(r).$$

   After discussion of the initial conditions (3.21) it should be possible to give arbitrary field configurations (especially mirror fields) by virtue of the boundary conditions $R(s_c,z) = R_c(z)$ for $R(s,z,t=0)$ and the condition $\frac{\partial}{\partial z} P(s) = 0$ (i.e. pressure constant along the field lines). This allows the end losses from theta pinches with magnetic mirrors to be investigated.

3) Application of the programme to the investigation of dynamic mirror fields in the linear theta pinch (LIMPUS with time and space varying mirrors).

Making allowance for finite electrical resistance ($\sigma < \infty$) and for radial compression (possibly including radial inertia terms) are further points that should be tackled once the problems stated have been solved.

The programmes for graphical representation of the results (functions of two space variables and time) were written in collaboration with U.Berkl. This was done for functions Y = Y(z,t) (see Figs. 8, 13,14,15) and also for functions Y = Y(r,z) (Fig.17). This graphical output could be handled not only by a curve plotter, but also by an electronic display unit (TV screen). The r-z diagrams were recorded from the screen with a camera at the G 3 computer at the Max-Planck-Institut für Physik und Astrophysik. In addition to the pictures shown here, a short film reproducing the dynamics (taking the density n = n(r,z) for running time t) as an example could also be made in this way.

References

[1] S. CHAPMAN, T.G. COWLING, "The Mathematical Theory of Non-Uniform Gases", University Press, Cambridge, 1964.

[2] F. HERTWECK, Allgemeine 13-Momenten-Näherung zur Fokker-Planck-Gleichung eines Plasmas, Laboratory Report IPP 6/1, Institut für Plasmaphysik, Garching b. München, 1962.

[3] A. SCHLÜTER, Dynamik des Plasmas, Z. Naturforsch. 5, 72 (1950).

[4] L. SPITZER, Jr., "Physics of Fully Ionized Gases", Interscience Publishers, New York - London, 1962.

[5] D. DÜCHS, Untersuchungen über den Einfluß von Neutralgas auf die Dynamik der Theta-Pinch-Entladungen, Laboratory Report IPP 1/14, IPP 6/10, Institut für Plasmaphysik, Garching b. München, 1963.

[6] H. FISSER, Numerical Solutions of the Magnetohydrodynamic Equations for One-Dimensional Theta-Pinch Geometry, (paper presented at the 'First European Conference on Controlled Fusion and Plasma Physics', Munich, 1966.

[7] K.V. ROBERTS, F. HERTWECK, S.J. ROBERTS, Thetatron, a Two-Dimensional Magnetohydrodynamic Computer-Programme, Laboratory Report CLM-R 29, Culham, 1963.

[8] K. HAIN, private communication.

[9] H. FISSER, private communication.

[10] D. DÜCHS, Two Dimensional Theta Pinch Dynamics with Transverse Magnetic Fields, Dissertation, Universität München, 1967.

[11] R.D. RICHTMYER, "Difference Methods of Initial-Value Problems", Interscience Publishers, Inc. New York, 1957.

[12] A. HEISS, H. HEROLD, E. UNSOELD, The Influence of Strong Magnetic Mirrors on Stability and Containment of a Theta-Pinch Plasma (Paper presented at the 'Topical Conference on Pulsed High Density Plasmas', Los Alamos, 1967).

[13] J.B. TAYLOR, J.A. WESSON, End Losses from a Theta-Pinch, Nucl. Fus. 5, 159 (1965).

[14] T.S. GREEN, D.L. FISHER, A.H. GABRIEL, F.J. MORGAN, A.A. NEWTON, Energy Loss from a Theta-Pinch, Phys. Fluids 10, 1663 (1967).